Edit (Levenstein) Distance...

Edit Distance

Edit Distance. Given two strings $S = s_1 \dots s_n$ and $T = t_1 \dots t_m$, the **edit distance** between S and T is defined as the *smallest number of atomic edit operations* necessary to transform S into T. The *atomic edit operations* are

- Character insertion. An insertion of a single character from the alphabet into any position in the string.
- Character deletion. A removal of any character from the string.
- Character replacement. A replacement of any character in the string with another character from the alphabet.

Example. Given a word `"cat"`, the following words have an edit distance of 1 from it:

- `"at"`, obtained from `"cat"` by deleting its first character:

```
cat
X||
_at
```

- `"cast"`, obtained from `"cat"` by inserting a character `"s"` into the third position of the string:

```
ca_t
||X|
cast
```

- `"vat"`, obtained from `"cat"` by replacing the first character with `"v"`:

```
cat
X||
vat
```

Computing the Edit Distance. We want to develop a dynamic programming algorithm for computing the edit distance. In preparation for this, we will consider using a data structure similar to the one we used when solving the LCS problem.

Let $c[i,j]$ be the edit distance between the prefixes $S_i = s_1 \dots s_i$ and $T_j = t_1 \dots t_j$ of the strings S and T. Our algorithm will construct the table $c[i,j]$. When completed, $c[n,m]$ will contain the edit distance between S and T.

The construction of $c[i,j]$ is guided by the following observations:

- $c[0,0] = 0$. For the sake of consistency, S_0 and T_0 are empty strings. The edit distance between two empty strings is 0.
- $c[0,j] = j$ for all $1 \leq j \leq m$. The edit distance between an empty string and any non-empty string of length j is j: the string can be constructed via j consecutive insertions.
- $c[i,0] = i$: see above (the empty string is constructed from $s_1 \dots s_i$ via i consecutive deletions).
- If $s_i = t_j$, then $c[i,j] = c[i-1,j-1]$. If the last characters of the two prefixes coincide, then the edit distance between them is the same as the edit distance between the prefixes without the last characters.
- If $s_i \neq t_j$, then an atomic edit is needed to match the last characters of the strings S_i and T_j. We must select one of the three possible atomic edits (insertion, deletion, or replacement). When selecting which one to use, we basically are reducing computing the edit distance between S_i and T_j to:
    1. computing the edit distance between S_{i-1} and T_{j-1} if replacement is chosen.
    2. computing the edit distance between S_{i-1} and T_j if deletion is chosen.
    3. computing the edit distance between S_i and T_{j-1} if insertion is chosen.

These insights can be properly encoded as follows:

$$c[i,j] = \begin{cases} 0 & \text{if } i = j = 0 \\ i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ c[i-1,j-1] & \text{if } i,j \geq 1 \text{ and } s_i = t_j \\ \min(c[i-1,j-1], c[i-1,j], c[i,j-1]) + 1 & \text{if } i,j \geq 1 \text{ and } s_i \neq t_j \end{cases}$$

Algorithm for Edit Distance Computation

Using the formula derived above, we can write the following algorithm for computing the table $c[i,j]$. The algorithm returns $c[n,m]$, which contains the edit distance between the input strings S and T.

```
Algorithm EditDistance(S = s_1 ... s_n, T = t_1 ... t_m)
begin
  declare c[0..n, 0..m];
  for i = 0 to n do
    c[i, 0] := 0;
  end for
  for j = 1 to m do
    c[0, j] := 0;
  end for
  for i = 1 to n do
    for j = 1 to m do
      if  s_i = t_j  then
        c[i, j] := c[i − 1, j − 1];
      else
        c[i, j] := min(c[i − 1, j], c[i, j − 1], c[i − 1, j − 1]) + 1;
      end if
    end for
  end for
  return c[n, m];
end
```

Analysis. The double nested loop executes $n \cdot m$ times. Each iteration runs in $O(1)$. Therefore, the algorithmic complexity of the EditDistance algorithm is $O(nm)$.