Mathematics for Computer Graphics - Barycentric Coordinates

Dr. Philippe B. Laval
Kennesaw State University

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Abstract

This document discusses barycentric coordinates and explains how they can be used to smoothly color a triangle when its vertices have different colors. The images which appear in this document are taken from [SP1]

1 Barycentric Coordinates

1.1 Calculus Review

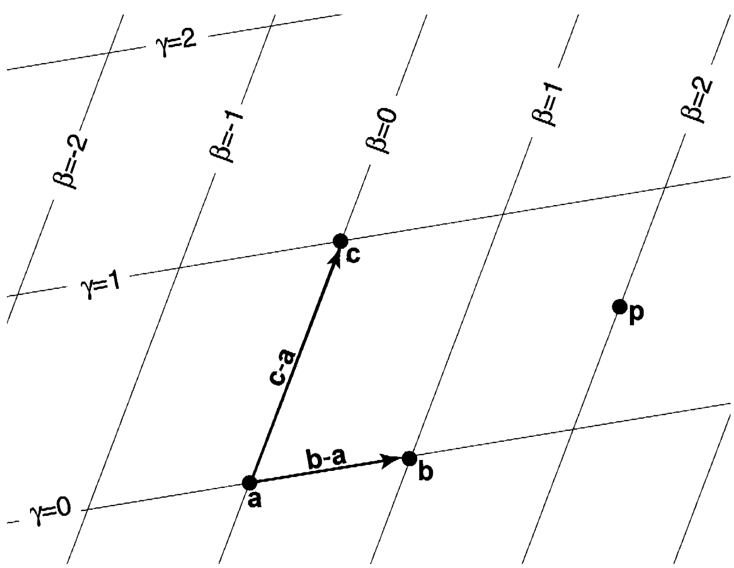

Figure 1: Barycentric Coordinates

The area of a 2D triangle whose vertices are $a = (x_a, y_a), b = (x_b, y_b), c = (x_c, y_c)$ (as shown in figure 1) is given by

$$Area = \frac{1}{2} \begin{vmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{vmatrix} \tag{1}$$

Note that this area is a "signed" area. In other words, it has a sign. To obtain the area the way we are used to define it, we take the absolute value. It is easy to implement this computation using our **vect3D** class. If we convert the 2D points in 3D points by adding 0 for their z-coordinate, then $\begin{vmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{vmatrix}$ will be the z-coordinate of $(b - a) \times (c - a)$.

If the vertices live in 3D space, the area of the corresponding triangle is

$$Area = \frac{1}{2} \|(b - a) \times (c - a)\| \tag{2}$$

This always gives a positive answer.

1.2 Barycentric Coordinates in 2D

Consider a 2D triangle whose vertices are $a = (x_a, y_a), b = (x_b, y_b), c = (x_c, y_c)$ (as shown in figure 1) and a point p. Barycentric coordinates allow us to express the coordinates of $p = (x, y)$ in terms of a, b, c. More specifically, the barycentric coordinates of p are the numbers β and γ such that

$$p = a + \beta (b - a) + \gamma (c - a) \tag{3}$$

If we regroup a, b and c, we obtain

$$\begin{aligned} p &= a + \beta b - \beta a + \gamma c - \gamma a \\ &= (1 - \beta - \gamma) a + \beta b + \gamma c \end{aligned}$$

Figure 1 shows the meaning of the numbers β and γ.

At this point, it is customary to define a third variable, α, by

$$\alpha = 1 - \beta - \gamma$$

We then have

$$p = \alpha a + \beta b + \gamma c$$

Definition 1 (Barycentric Coordinates) *The barycentric coordinates of the point p in terms of the points a, b, c are the numbers α, β, γ such that*

$$p = \alpha a + \beta b + \gamma c \tag{4}$$

with the constraint

$$\alpha + \beta + \gamma = 1 \tag{5}$$

We first look at some of the properties of the barycentric coordinates.

Barycentric coordinate are defined for all points in the plane. They have several nice features:

1. A point p is inside the triangle defined by a, b, c if and only if

$$\begin{aligned} 0 &< \alpha < 1 \\ 0 &< \beta < 1 \\ 0 &< \gamma < 1 \end{aligned}$$

   This is very important. It provides an easy way to test if a point is inside a triangle.

2. If one of the barycentric coordinates is 0 and the other two are between 0 and 1, the corresponding point p is on one of the edges of the triangle.

3. If two of the barycentric coordinates are zero and the third is 1, the point p is at one of the vertices of the triangle.

4. By changing the values of α, β, γ between 0 and 1, the point p will move smoothly inside the triangle. This can (and will) be applied to other properties of the vertices such as color.

5. The center of the triangle is obtained when $\alpha = \beta = \gamma = \frac{1}{3}$. If the triangle is made of a certain substance which is evenly distributed throughout the triangle, then these values of α, β, γ would give us the center of gravity.

1.2.1 Finding α, β, γ

Let us first notice that because of equation 5, it is enough to find two of the three barycentric coordinates.

One way of finding α, β, γ is to write equation 3 in terms of the coordinates of the various points involved. This gives us the following system

$$\left\{ \begin{array}{c} x = x_a + \beta (x_b - x_a) + \gamma (x_c - x_a) \\ y = y_a + \beta (y_b - y_a) + \gamma (y_c - y_a) \end{array} \right. \tag{6}$$

which can be solved using your favorite method.

We can also use geometry to find α, β, γ. Let A_a, A_b and A_c be as in figure 2 and let A denote the area of the triangle. Also note that the point inside the triangle on figure 2 is the point we called p. Consider the triangles in figure 3. These are different triangles drawn for a fixed value of β. They have the same area since they have the same base and height. This area was denoted A_b on figure 2. Thus, we see that A_b only depends on β. Therefore, we have

$$A_b = C\beta \tag{7}$$

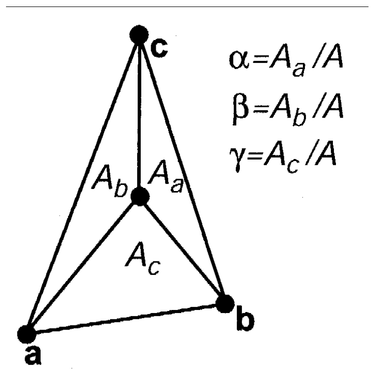

Figure 2: Barycentric Coordinates as a Ratio of Areas

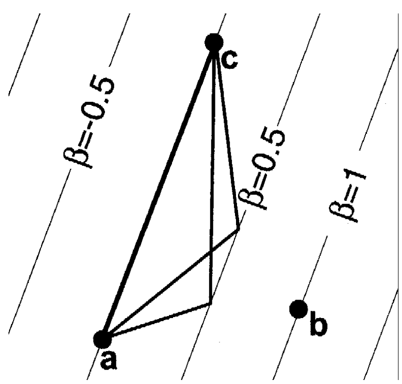

Figure 3: A_b Only Depends on β

for some constant C. When p is on b that is when $\beta = 1$, we have $A_b = A$. Equation 7 gives us $A = C$. Therefore we see that

$$\beta = \frac{A_b}{A}$$

Similarly, we have

$$\gamma = \frac{A_c}{A}$$

$$\alpha = \frac{A_a}{A}$$

These various areas can then be computed using formula 1. More specifically,

$$\beta = \frac{\begin{vmatrix} x_a - x_c & x - x_c \\ y_a - y_c & y - y_c \end{vmatrix}}{\begin{vmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{vmatrix}}$$

$$\gamma = \frac{\begin{vmatrix} x_b - x_a & x - x_a \\ y_a - y_a & y - y_a \end{vmatrix}}{\begin{vmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{vmatrix}}$$

$$\alpha = 1 - \beta - \gamma$$

1.3 Application of Barycentric Coordinates to Coloring Triangles

Let us assume that we are using the RGB model. That is all colors can be obtained by mixing R (red), G (green) and B (blue). Usually, with such a model, the level of each color channel is a number between 0 and 255. To specify the color at any point, we must specify a triplet (R, G, B) where R, G, and B are integers between 0 and 255. They indicate how much of red, green and blue is used in the color.

When using Java, there is a built-in class to handle colors. It is called **Color**. This class has some built-in predefined colors. Here are some examples:

Color.black, Color.blue, Color.cyan, Color.gray, Color.green, Color.magenta, Color.orange, Color.pink, Color.red, Color,white, Color.yellow. To get any other color, one uses a statement such as **new Color(R,G,B)** where **R,G,B** are integers between 0 and 255.

It is also possible to use a single integer to represent colors. Keeping in mind that an integer has 32 bits, bits $0 - 7$ contain the **R** level, bits $8 - 15$ contain the **G** level and bits $16 - 23$ contain the **B** level. The remaining bits are unused and set to 0.

Let us see now how barycentric coordinates can be used to smoothly color a triangle, given the color of its vertices. Using the notation above, let us assume that C_a is the color of a, C_b is the color of b and C_c is the color of c. Each color is in fact a triplet. We will use the notation $C_a = (R_a, G_a, B_c)$ and similar notation for the remaining points. We would like to color the triangle so that there is a smooth coloring throughout the triangle. We use the fact that by changing the values of α, β, γ between 0 and 1, the point $p = \alpha a + \beta b + \gamma c$ will move smoothly inside the triangle. In other words, small changes in α, β, γ will result in small changes in the location of p. We apply this to colors. We let

$$C = \alpha C_a + \beta C_b + \gamma C_c \tag{8}$$

(we really do this for every color channel). Small changes in α, β, γ will result in small changes in the color. Therefore, the color will change smoothly as we move within the triangle. To color smoothly a triangle given the color of its vertices, we can use the following algorithm:

1. For each point $P = (x, y)$ inside the triangle, find α, β, γ
2. Use α, β, γ to interpolate the color of the point from the color of the vertices using relation 8.
3. Plot the point with coordinates (x, y) and color computed above.

There is an applet on the web site for the class to illustrate this. It allows the user to draw a rectangle and specify the color of the vertices. Clicking inside the triangle will result in the coordinates (Cartesian and barycentric) of the point where the user clicked to be displayed. If Coloring of the triangle is checked, the RGB color of the point will also be displayed.

1.4 Barycentric Coordinates in 3D

We use the same notation as in the 2D case. The only difference is that now points have three coordinates. So, we have $a = (x_a, y_a, z_a)$, $b = (x_b, y_b, z_b)$ and $c = (x_c, y_c, z_c)$. Barycentric coordinates extend naturally to 3D triangles and they have the same properties. In other words, we have:

Definition 2 (Barycentric Coordinates) *The barycentric coordinates of the point p in terms of the points a, b, c are the numbers α, β, γ such that*

$$p = \alpha a + \beta b + \gamma c \tag{9}$$

with the constraint

$$\alpha + \beta + \gamma = 1 \tag{10}$$

Barycentric coordinate are defined for all points in the plane. They have several nice features:

1. A point p is inside the triangle defined by a, b, c if and only if

$$\begin{aligned} 0 &< \alpha < 1 \\ 0 &< \beta < 1 \\ 0 &< \gamma < 1 \end{aligned}$$

   This is very important. It provides an easy way to test if a point is inside a triangle.

2. If one of the barycentric coordinates is 0 and the other two are between 0 and 1, the corresponding point p is on one of the edges of the triangle.

3. If two of the barycentric coordinates are zero and the third is 1, the point p is at one of the vertices of the triangle.

4. By changing the values of α, β, γ between 0 and 1, the point p will move smoothly inside the triangle. This can (and will) be applied to other properties of the vertices such as color.

5. The center of the triangle is obtained when $\alpha = \beta = \gamma = \frac{1}{3}$. If the triangle is made of a certain substance which is evenly distributed throughout the triangle, then these values of α, β, γ would give us the center of gravity.

What differs from the 2D case is how they are computed. α is still related to the ratio $\frac{A_a}{A}$. There is a similar relation for β and γ. However, in the 2D case, the formula we used for the area gave us a "signed" area. It depended on the orientation of the orientation of the triangle. The formula for the 3D area of a triangle, as noticed above, is always positive. To also make it a sign area, we also need to take a dot product. More specifically, we define the following quantities:

- n is the normal to the triangle T with vertices (a, b, c) in counterclockwise order. In other words, $n = (b - a) \times (c - a)$.
- n_a is the normal to T_a, the triangle with area A_a as shown in figure 2. $T_a = (b, c, p)$ in counterclockwise order. Thus, $n_a = (c - b) \times (p - b)$.
- n_b is the normal to T_b, the triangle with area A_b as shown in figure 2. $T_b = (c, a, p)$ in counterclockwise order. Thus, $n_b = (a - c) \times (p - c)$.
- n_c is the normal to T_c, the triangle with area A_c as shown in figure 2. $T_c = (a, b, p)$ in counterclockwise order. Thus, $n_c = (b - a) \times (p - a)$.

The quantity $\frac{n.n_a}{\|n\|\|n_a\|} = 1$ if p is inside T, -1 otherwise. The same is true for $\frac{n.n_b}{\|n\|\|n_b\|}$ and $\frac{n.n_c}{\|n\|\|n_c\|}$. Multiplying $\frac{A_a}{A}$ by $\frac{n.n_a}{\|n\|\|n_a\|}$ will then give us a signed area, depending on whether p is inside T or outside. Since $A_a = \|n_a\|$ and $A = \|n\|$, we have:

$$\frac{A_a}{A}\frac{n.n_a}{\|n\|\,\|n_a\|} = \frac{\|n_a\|}{\|n\|}\frac{n.n_a}{\|n\|\,\|n_a\|} = \frac{n.n_a}{\|n\|^2}$$

We obtain similar formulas for the other ratios. Thus, in the case of a 3D triangle, we can define the barycentric coordinates by:

$$\begin{aligned} \alpha &= \frac{n.n_a}{\|n\|^2} \\ \beta &= \frac{n.n_b}{\|n\|^2} \\ \gamma &= \frac{n.n_c}{\|n\|^2} \end{aligned} \tag{11}$$

2 Assignment

1. Solve the system in equation 6 and find α, β, γ.
2. Verify that the second method shown in the text gives us the same answer.
3. Prove that if R, G, and B are integers between 0 and 255, then $\alpha R + \beta G + \gamma B$ is also between 0 and 255. This is important. It means we can mix RGB colors using barycentric coordinates, the values we obtain are still within the allowable range.
4. Explain how you would convert colors between the RGB format and the single integer format.

3 Resources

This is a list of books and other resources I used to compile these notes.

References

[BG1] Burger, Peter, and Gillies, Duncan, *Interactive Computer Graphics*, Addison-Wesley, 1990.

[DD1] Deitel, H.M., and Deitel, P. J., Java, How to Program, Prentice Hall, 1999.

[DP1] Dunn, Fletcher and Parberry, Ian, *3D Math Primer for Graphics and Game Development*, Wordware Publishing, Inc., 2002.

[FD1] Foley, J.D., Van Dam, A., Feiner, S.K., and Hughes, J.F., Computer Graphics, Principles and Practices, Addison-Wesley, 1995.

[FD2] Foley, J.D., Van Dam, A., Feiner, S.K., Hughes, J.F., and Philipps, R.L., *Introduction to Computer Graphics*, Addison-Wesley, 1997.

[H1] Hill, F.S. JR., *Computer Graphics Using Open GL*, Prentice Hall, 2001.

[LE1] Lengyel, Eric, *Mathematics for 3D Game Programming & Computer Graphics*, Charles River Media, Inc., 2002.

[SH1] Schildt, Herbert, *Java2, The Complete Reference*, McGraw-Hill, 2001

[SE1] Schneider, Philip J., and Eberly, David H., *Geometric Tools for Computer Graphics*, Morgan Kaufman, 2003.

[SP1] Shirley, Peter, *Fundamentals of Computer Graphics*, A K Peters, 2002.

[SJ1] Stewart, James, *Calculus, Concepts and Contexts*, second edition, Brooks/Cole, 2001.

[WG1] Wall, David, and Griffith, Arthur, *Graphics Programming with JFC*, Wiley, 1999.

[AW1] Watt, Alan, *3D Computer Graphics*, Addison-Wesley, 2000.

[AW2] Watt, Alan, *The Computer Image*, Addison-Wesley, 1999.