Greedy Algorithms: Theory

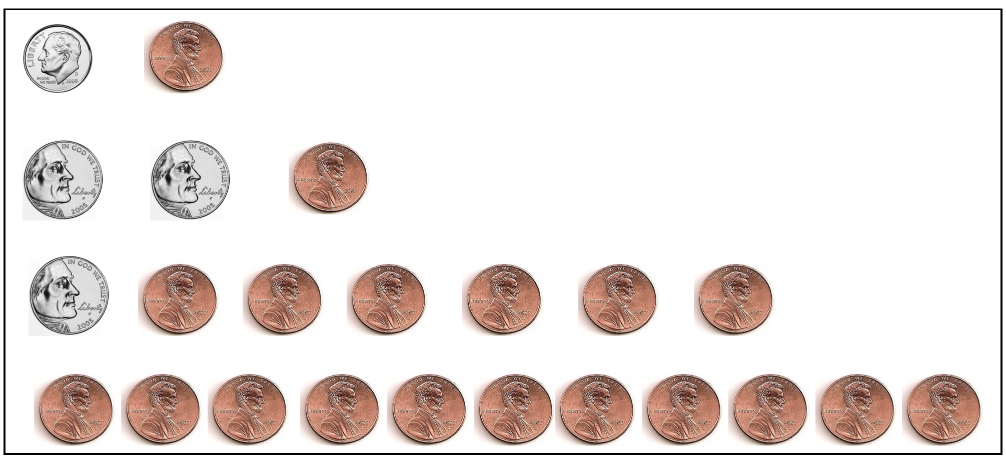
Optimization Problems

Greedy algorithms and dynamic programming algorithms[1] are usually designed for a special class of problems called **optimization problems**.

Optimization Problem. A computational problem in which the object is to find *the best of all possible solutions* is called an ***optimization problem.***

Typically, an optimization problem comes with an ***objective function*** which maps *solutions* to the problem to a set of values. ***An optimal solution*** is a solution that either maximizes or minimizes (depending on the requirements of the problem) the *objective function*.

Example. Making change problem is an optimization problem. Given N, the amount of money to give in change, the problem space consists of a *potentially* large number of possible solutions: any collection of coins that adds up to N is a possibility. For, example, here are all possible ways in which one can make 11 cents in change:



[1] To be studied next.

The objective function evaluating the quality of a solution $Sol = \{coin_1, \ldots, coin_s\}$:

$$val(Sol) = |Sol| = s.$$

For the Making Change problem, *an optimal solution* ***minimizes*** the value of the objective function $val()$.

Properties of Optimization Problems

Problems solvable using both dynamic programming and greedy algorithms exhibit the **optimal substructure property**.

Additionally, problems solving using greedy algorithms exhibit the **greedy choice property**.

Optimal Substructure property.

Subproblems. Given a computational problem P with input I, a subproblem of P_s of P is any computational problem whose input $I_s \subset I$ and whose desired output should satisfy the same properties as the output of P.

Example. Consider an instance P of the Making Change problem, asking us for the best way to change 45 cents. A request to make change for any sum that is *smaller* than 45 cents[2], e.g., 36 cents, is a *subproblem* of P.

Problem decomposition. A problem is called *decomposable* if it can be solved, by solving a number of its subproblems.

Optimal Substructure property. A problem is said to have an **optimal substructure property** iff its *optimal solution contains within it* ***optimal solutions to its subproblems***.

Theorem 1. Problem Activity Selection has **optimal substructure property.**

Proof. Consider an instance P of the Activity Selection problem with input $A[1..M][1..2]$. Let set $S = \{j_1, \ldots j_k\}$ $(k < M)$ be an optimal solution to P. (Here, $j_1, \ldots j_k$ are indexes of the activities from $A[][]$ that are being scheduled.)

Consider some activity $A[j]$ where $j \in S$. $A[j]$ splits the list of all activities $A[]$ into three parts:

1. $A^{early} = \{A[i] \in A[] | A[i][2] < A[j][1]\}$: activities that end before $A[j]$ starts.
2. $A^{conflict} = \{A[i] \in A[] | [A[i][1], A[i][2])\ subseteq\ [A[][1], A[j][2]) \neq \emptyset$: activities that conflict with $A[j]$.

[2] Our definition uses $I_s \subset I$ relation between the inputs of a problem and its subproblem. In case of the Making Change problem, we can view input rendered in a unary alphabet of bits. The set of bits needed to render 45 will be a superset to the set of bits needed to render any smaller number of cents.

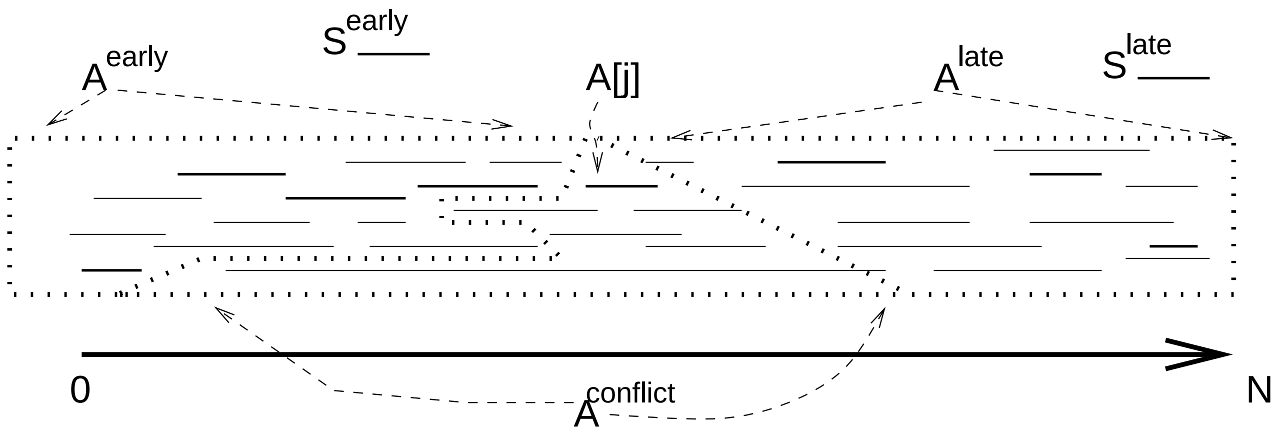


Figure 1: Optimal substructure for the Activity Selection Problem (Theorem 2).

3. $A^{late} = \{A[i] \in A[]|A[i][1] > A[j][2]\}$: activities that start after $A[j]$ ends.

Similarly, $A[j]$ partitions the optimal solution S into three parts:

1. $S^{early} = \{i \in S|A[i][2] < A[j][1]\}$: activities that end before $A[j]$ starts.

2. $\{j\}$.

3. $S^{late} = \{i \in S|A[i][1] > A[j][2]\}$: activities that start after $A[j]$ ends.

The structure of the problem is shown in Figure 1. The set S of optimally scheduled activities is shown in bold. The partitions of A and S are circled by dotted lines.

We claim that:

- S^{early} is the **optimal solution** for the Activity Selection problem with input A^{early};

- S^{late} is the **optimal solution** for the Activity Selection problem with input A^{late};

We prove the first statement. The proof of the second is symmetric.

Suppose S^{early} is **NOT** an optimal solution for the Activity Selection problem with inbput A^{early}. Consider then a set $S' = \{f_1, \ldots, f_l\}$, which **is** an optimal solution for this problem. We know that $val(S') = |S'| > |S^{early}| = val(S^{early})$. But then, S cannot be an optimal solution for original problem, because the following set

$$S^* = S' \cup \{j\} \cup S^{late}$$

is an optimal solution and $val(S^*) < val(S)$.

Indeed, S^* is a *solution* for the Activity Selection problem, because no $A[i]$, such that $i \in S'$ conflicts with $A[j]$ or with any $A[j']$ where $j' \in S^{late}$. And because $|S'| < |S^{early}|$, $|S^*| > |S|$, and therefore $|S|$ cannot be an optimal solution, which contradicts our original assumption.

Greedy Choice Property

Greedy Choice property. An optimization problem has **greedy choice property** if

- It has *optimal substructure property.*
- The optimal solution of a problem can be constructed from an optimal solution to a subproblem by extending the solution of the subproblem *in a locally optimal way.*

Locally optimal way. Consider a problem P with solution S. We want to exted the solution of P to a superproblem P'. Consider a list of alternatives $A_1, \ldots, A_k$, such that $S \cup A_i$ is a feasible (albeit not necessarily optimal) solution of P'. Consider a *local quality* function $q(A_i)$, which supplies for each possible choice A_i a *local goodness* estimate.

A *locally optimal way* of selecting a solution for P' is a solution $S \cup A_i$, such that $q(A_i)$ is optimal[3].

Philosophy of Greedy Choice

The key feature of the greedy choice property is the fact that at each step we add into our solution an element with the *best local quality* estimate. This makes greedy algorithms **very efficient**, because only one selection needs to be made per algorithm iteration, and that selection is never "questioned".

Greedy Choice Property Theorems

Theorem 2. Problem Activity Selection has **greedy choice property**.

Proof. Consider an instance of the Activity Selection problem with input $A[1..M][1..2]$. Let A^t be a subproblem of this Activity Selection problem containing only activities $A[i]$ that start **after time** t.

Further, Let $A[k] \in A^t$ be the activity with the *earliest finishing time*, i.e., $A[k][2] = \min_{A[k] \in A^t}(A[j][2])$. We will prove that $A[k]$ is included in some *optimal solution* for A^t.

Let S^t be an *optimal solution* for the Activity Selection problem for A^t. Consider an element $A[i] \in S^t$ with the earliest finish time. We consider two cases:

- **Case 1:** $A[i] = A[k]$, i.e., $A[k] \in S^t$.
- **Case 2:** $i \neq k$. Because $A[k]$ ends earlier than $A[i]$, it follows, that $A[i] \notin S^t$. Consider now a set $S^t_k = S^t - \{A[i]\} \cup \{A[k]\}$.

  We claim that

  (a) $|S^t_k| = |S^t|$. This is indeed so, as S^t_k is formed from S^t by replacing **one** activity with another **single** activity.

[3] Minimal or maximal, depending on the nature of the problem.

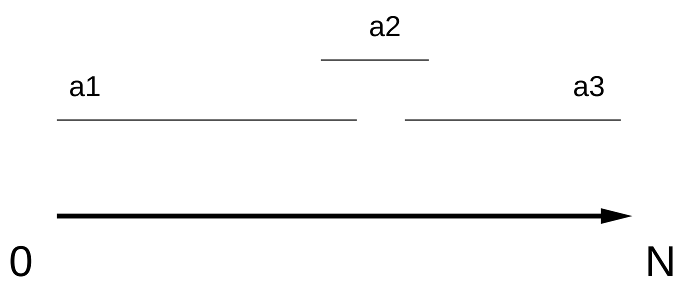


Figure 2: Selection of shortest activity is not an optimal greedy strategy for the Activity Selection problem.

(b) S_k^t is **conflict-free**. Indeed, $S^t - \{A[i]\}$ contains no activities that conflict with $A[i]$ (as S^t is an optimal solution for A^t). But $A[k][2] < A[i][2]$ (i.e., A[k] will finish earlier than A[i]). Therefore, no activity from $S^t - \{A[i]\}$ can possibly conflict with $A[k]$. But then, S_k^t contains **no conflicts**.

From these two statements we obtain that S_k^t is an **optimal solution** for A^t.

Theorem 3. Problem Fractional Knapsack has **greedy choice property**.

(left as exercise)

Greedy Algorithm Design

Greedy algorithms can be designed for virtually any optimization problem. They can be:

- optimal for problems that have *optimal substructure property* and *greedy choice property.*
- approximations for problems that have only *optimal substructure property.*

For some problems greedy algorithms give good approximations. For some other problems, greedy algorithms can be really really bad (sometimes, can lead to worst solutions).

Design. Given a problem, establish the following:

1. Proper objective function for the problem.
2. Optimal substructure property for the problem.
3. Proper *quality function* for evaluating solution steps (i.e., your greedy approach).
4. Greedy choice property.

Note, that greedy choice property can only be established **after** you select your greedy strategy, as computational problems admit optimal greedy algorithms for *some* greedy approaches, but not for others.

Example. Consider two greedy approaches to the Activity Selection problem.

- Select the activity that finishes the earliest.
  - **Theorem 2** shows that Activity Selection has greedy choice property w.r.t. this strategy.
- Select the shortest activity.
  - Figure 2 shows that Activity Selection with this strategy does not have greedy choice property. There, the optimal solution is the activity set $\{a1, a3\}$, while selecting the shortest activity will lead to the solution $\{a2\}$, which is not optimal.

Greedy algorithm outline. Once you either:

- establish that an optimal greedy algorithm exists for a problem, **or**
- decide that a greedy *approximation algorithm* is good enough for you

you need to devise the algorithm itself.

Greedy algorithms can have the following components:

- Selection function. On each step, this function, given the currently constructed part of the solution, and current choices for including into the solution, **will make the greedy choice.**
- Feasibility function. On each step, this function indicates whether the currently constructed part of the solution is *feasible*, i.e., can lead to a solution.
- Pruning function. On each step, once the greedy choice is made, this function will eliminate all possibilities that conflict with the current portion of the solution.
- Objective function. This function evaluates the current solution or partial solution and returns a numeric value for it.
- Solution function. This function evaluates current partial solution to test if it is, in fact, a complete solution.

The overall organization of a greedy algorithm is:

```
Initialization Step: Set solution S ← ∅;
Solution Step: while S is feasible and S is NOT a solution do
   Establish space of choices C;
   Update solution: S ← S∪ Greedy_Select(C);
endif
if S is NOT feasible return FAIL
else return S;
```