Data 401
Feature Engineering

Dennis Sun

October 3, 2016

1. Review of Last Time

2. Basis Expansions

3. Interactions

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- Categorical variables need to be expanded into a bunch of binary variables.

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(`np.linalg.cond(A)` in Numpy.)

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This feature engineering is necessary to get good prediction accuracy. In this lecture, we explore some techniques.

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City MPG vs. Price

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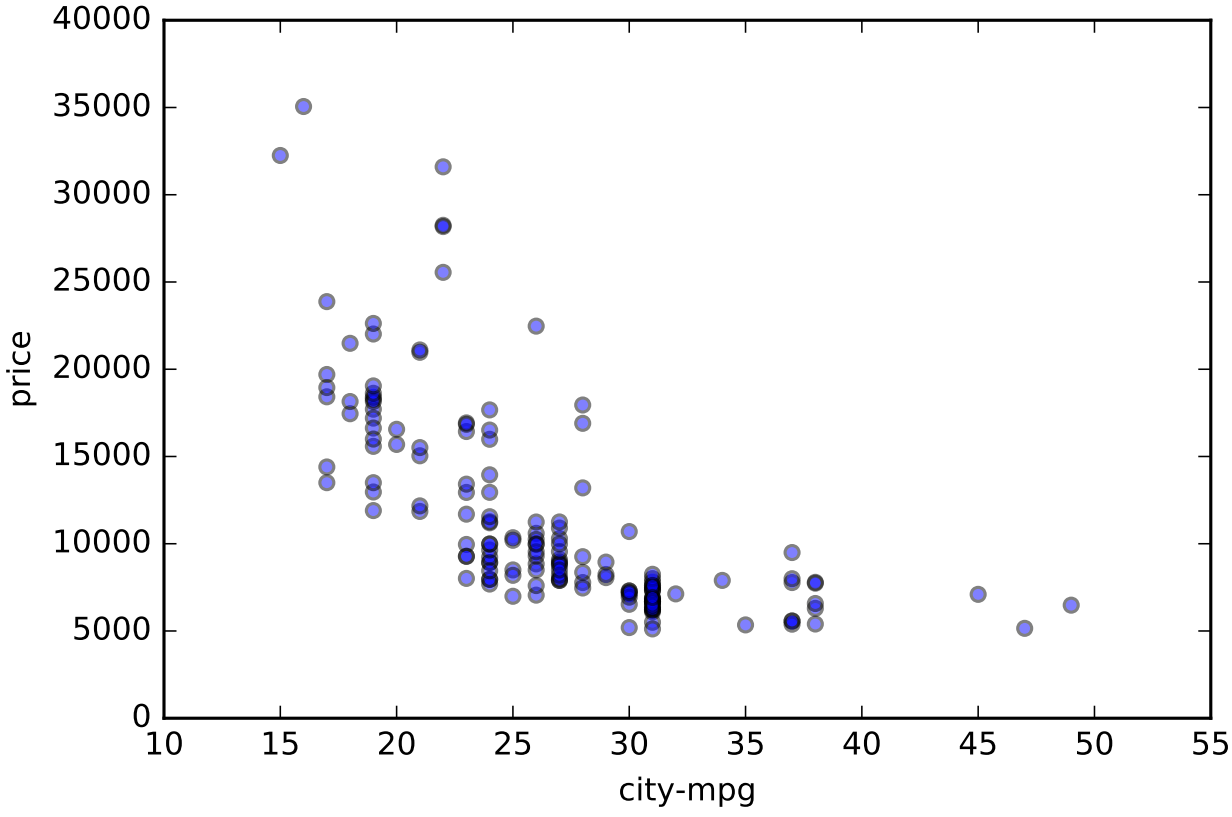

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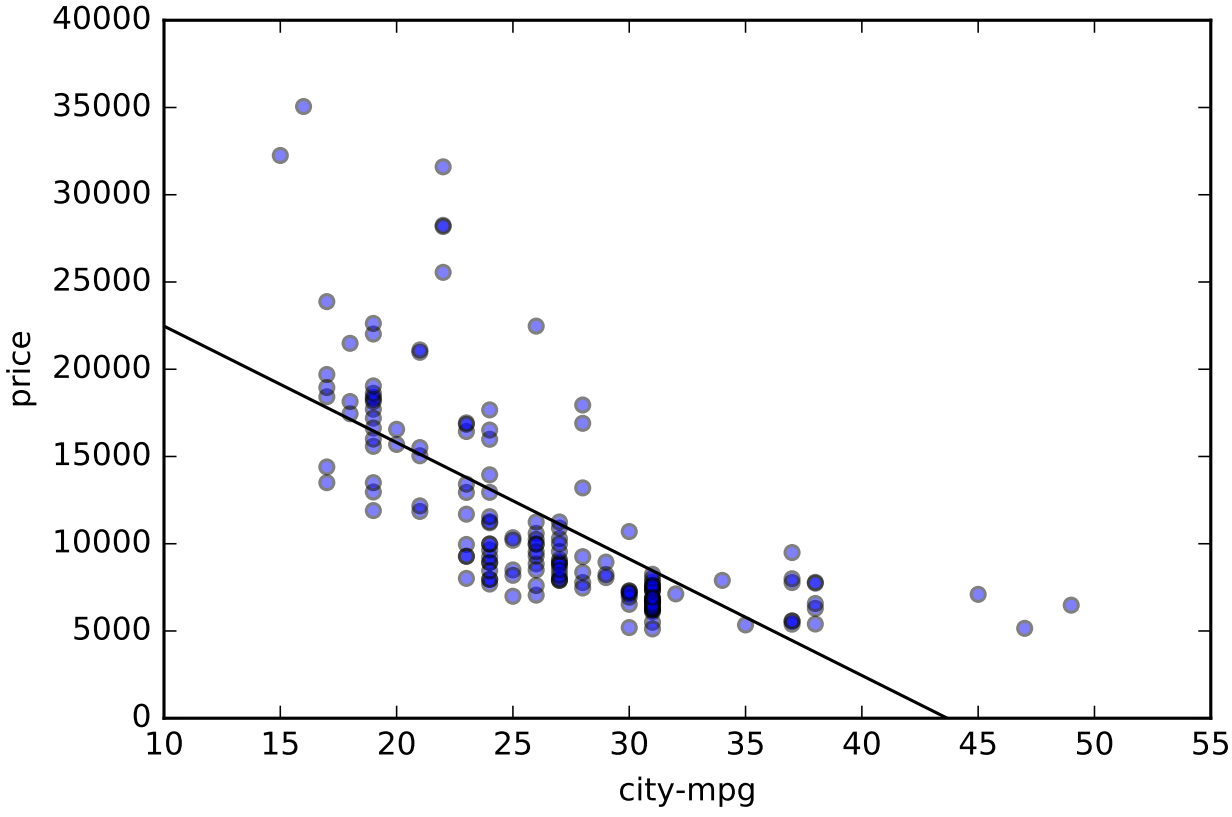

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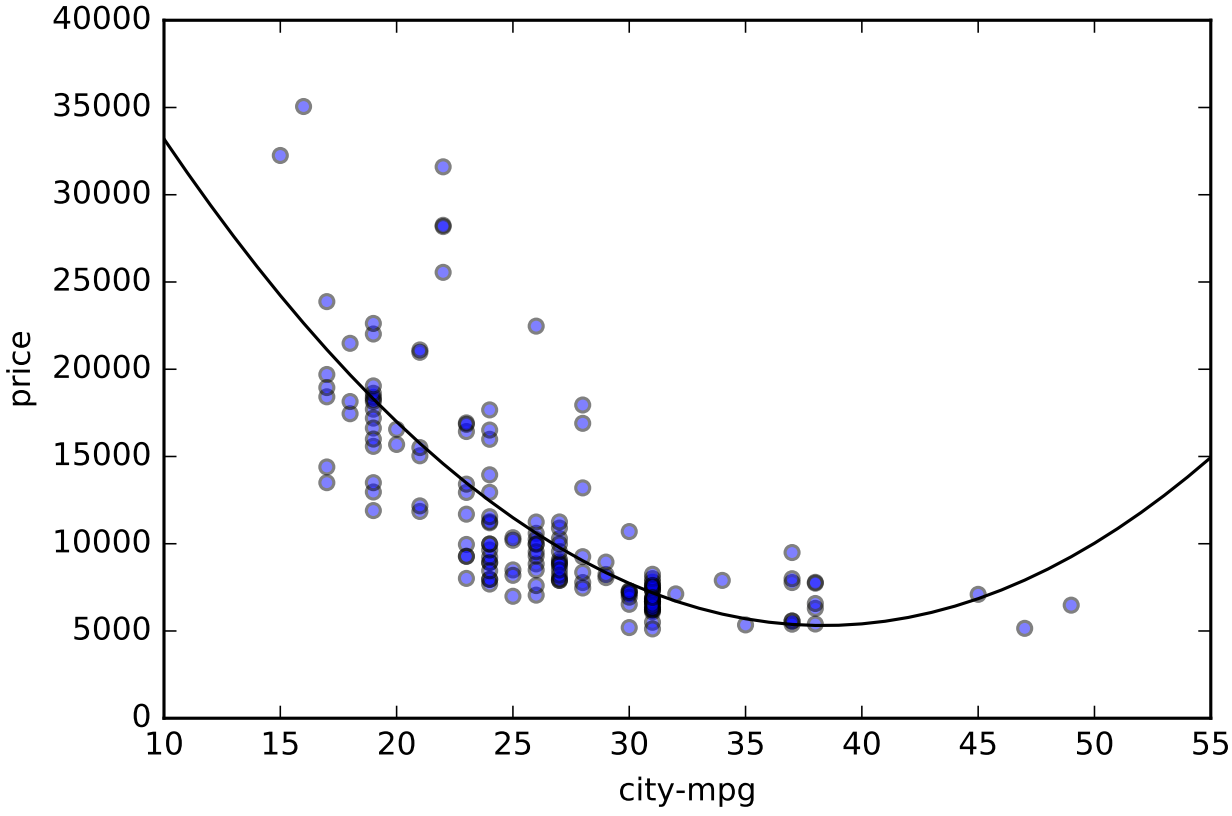

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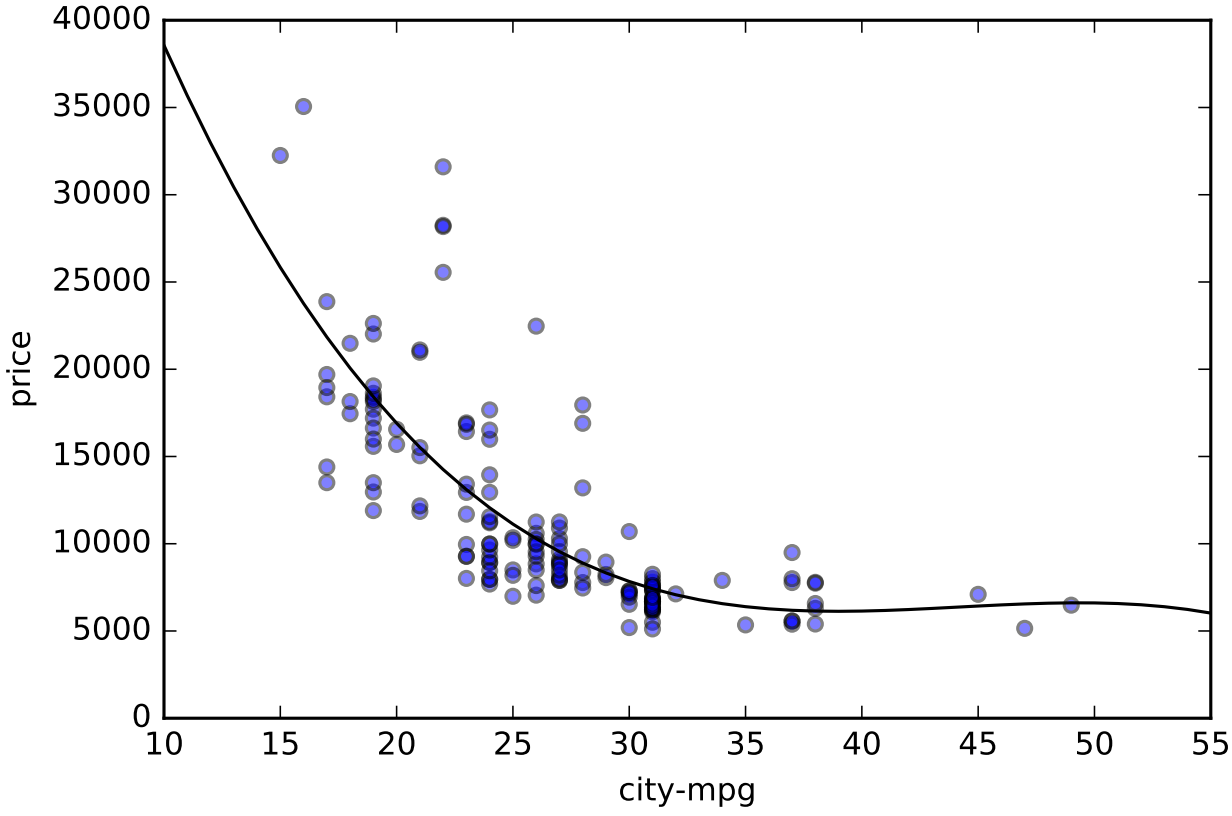

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So is the optimal strategy to use as many terms as possible?

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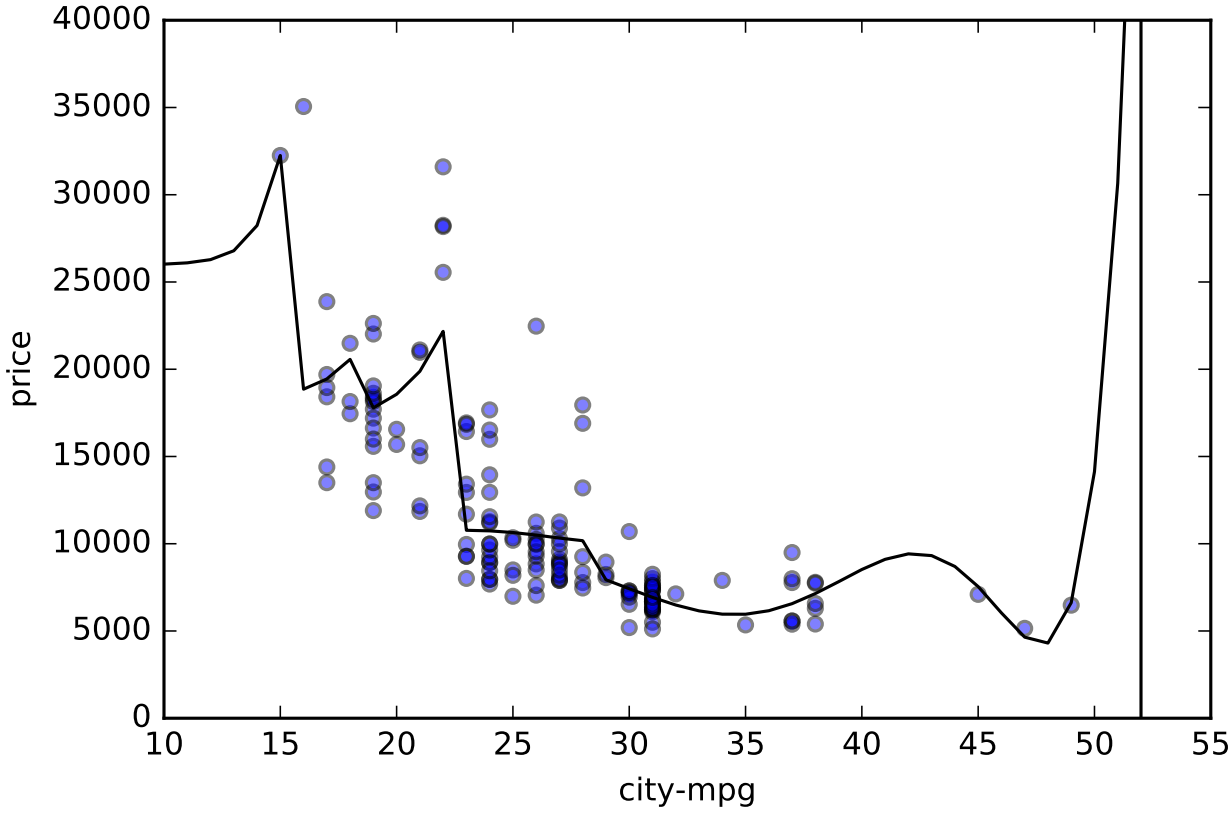

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + ... + \beta_{19} x^{19}$$

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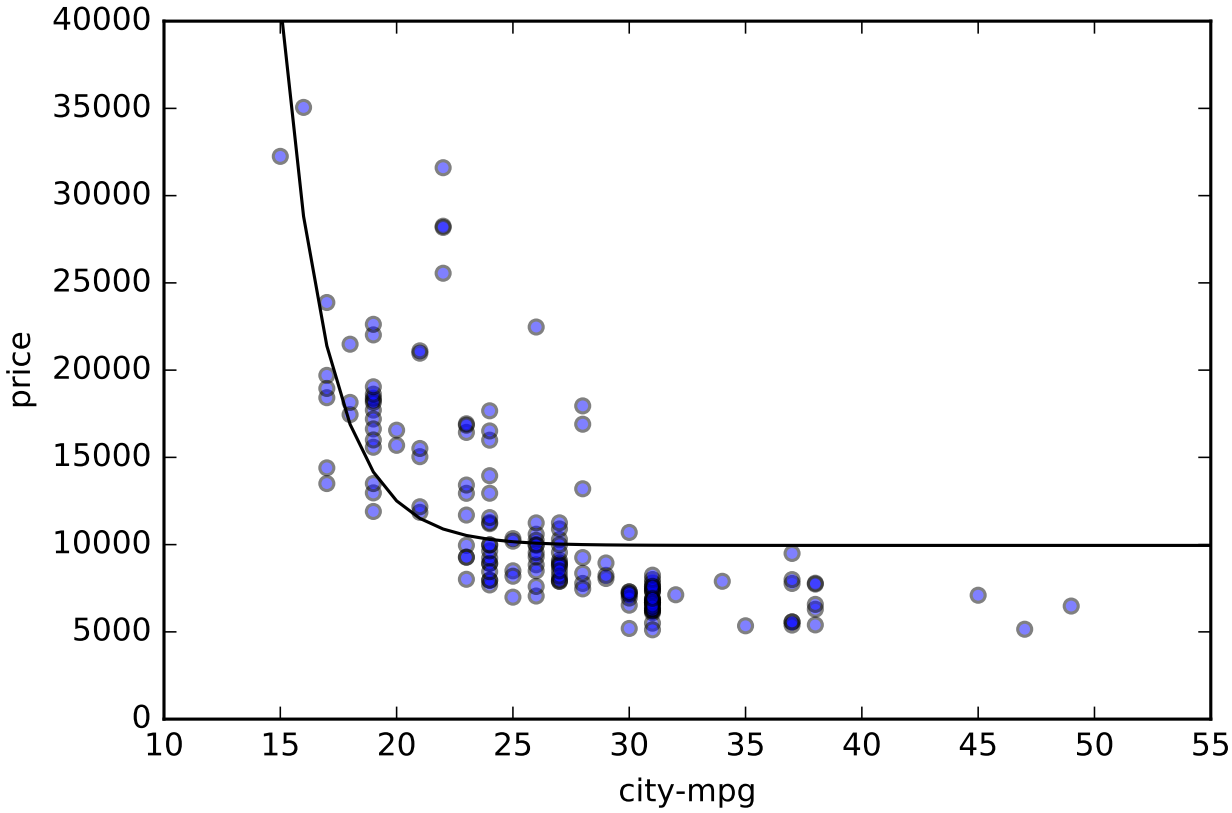
This fundamental tradeoff is called the **bias-variance tradeoff**. We want to use a sufficiently complex model to reduce bias, but we don't want a model so complex that we incur a variance penalty.

Other Basis Expansions

We don't have to use polynomials!



$$y = \beta_0 + \beta_1 \exp(-.5x)$$

Other Basis Expansions

In general, we can choose any functions f_1, f_2, ..., f_k:

$$y = \beta_0 + f_1(x) + f_2(x) + ... + f_k(x).$$

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Common choices are polynomials and spline basis functions.

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$$X = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 \end{pmatrix}$$

Basis Expansions in Scikit-Learn

```
from sklearn.linear_model import LinearRegression
for i in range(1, p+1):
  data["city-mpg^%d" % i] = data["city-mpg"] ** i

model = LinearRegression()
model.fit(data[["city-mpg^%d" % i for i in range(1, p+1)]], data["price

# plot the line
x_test = np.arange(10, 60)
y_test = model.intercept_
for i in range(1, p+1):
  y_test += model.coef_[i-1] * x_test ** i
plt.plot(x_test, y_test)
```

Basis Expansions in Code

In-Class Exercise

Open the notebook `Implementing Linear Regression.ipynb`. *Pick a different quantitative predictor in the autos data set and fit a curve to it. You should try a line first, then using basis expansion if that is unsatisfactory.*

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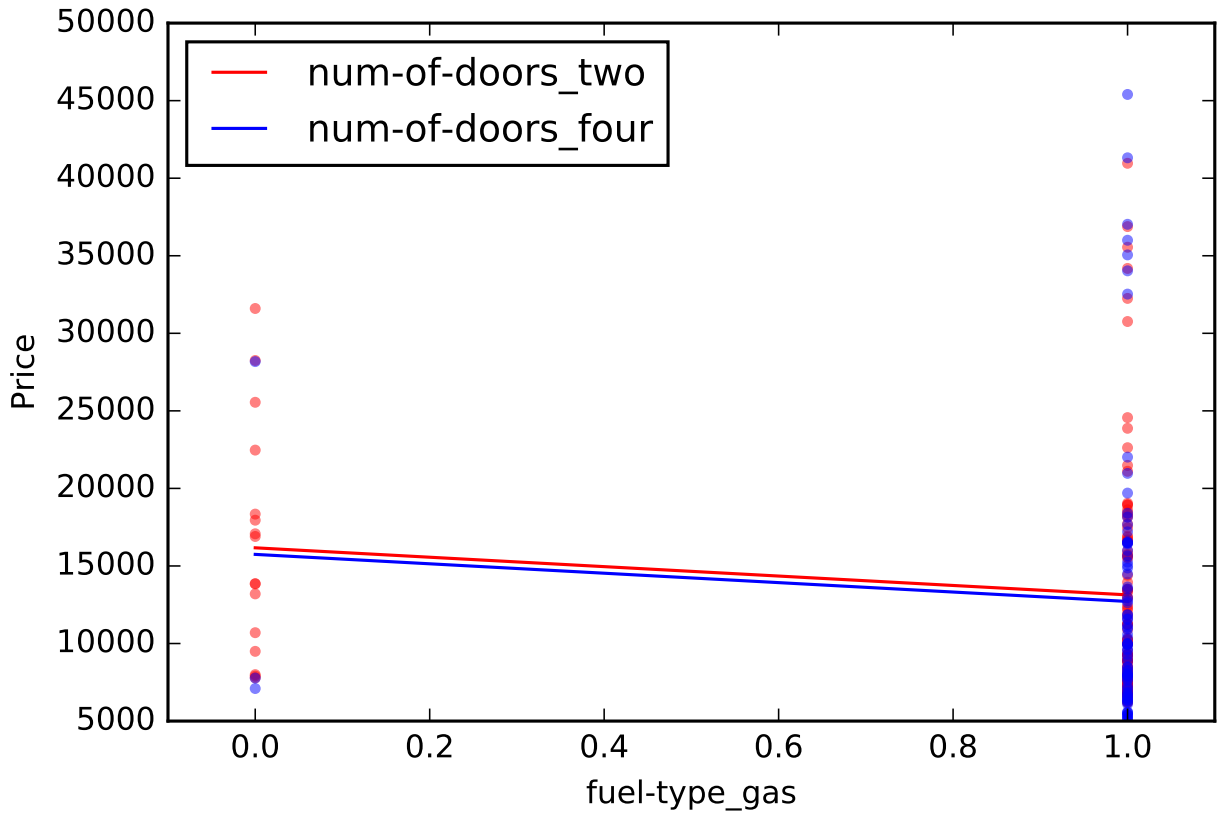
Let's consider the case where x_1 and x_2 are both binary variables, e.g., fuel-type and engine-location.

No Interaction Model

No interaction:

$$y = \beta_0 + \beta_1 x_{\text{fuel-type\_gas}} + \beta_2 x_{\text{num-of-doors\_four}}$$



With Interaction Model

With interaction:

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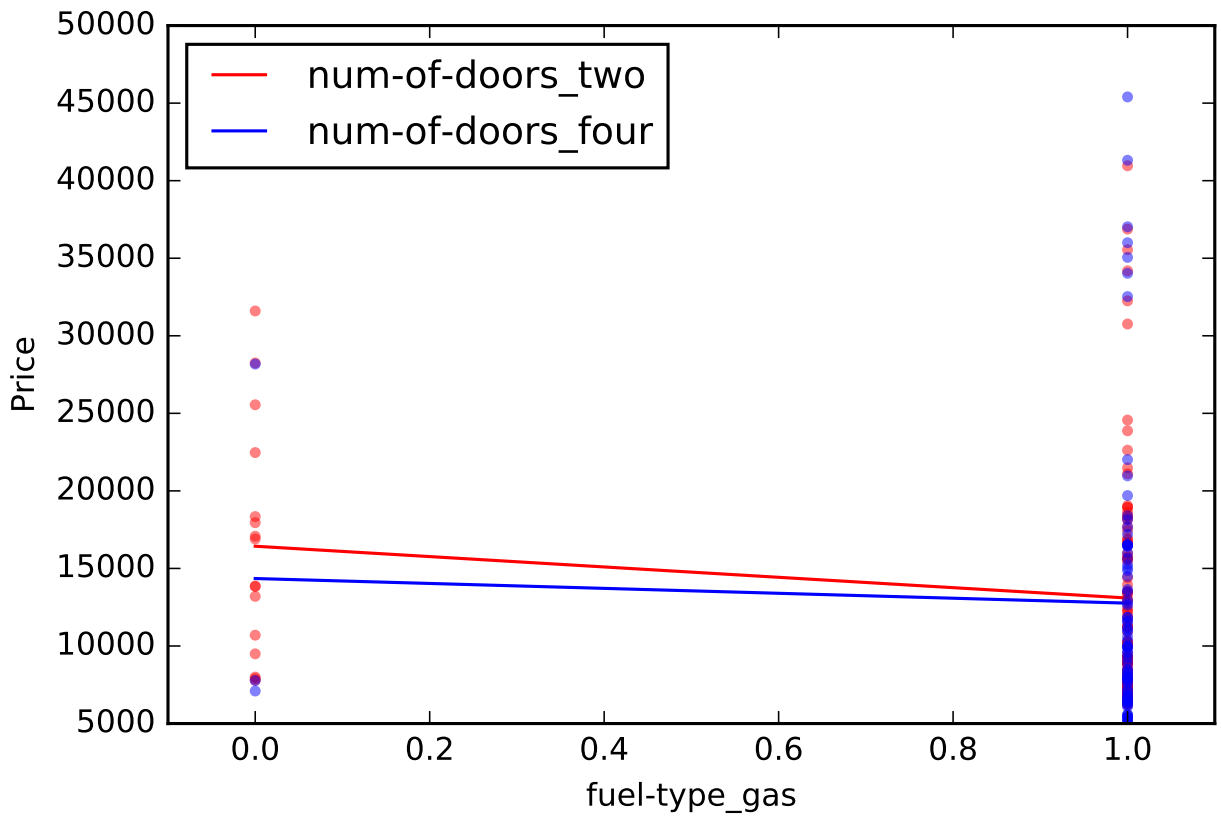

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$$\begin{aligned} y = \beta_0 + \beta_1 x_{bmw} + ... + \beta_{k-1} x_{volvo} + \\ \alpha_1 x_{sedan} + ... + \alpha_{\ell-1} x_{convertible} + \\ (\alpha\beta)_{11} x_{bmw} x_{sedan} + ... + (\alpha\beta)_{\ell-1,k-1} x_{volvo} x_{convertible} \end{aligned}$$

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What are some advantages and disadvantages of fitting an interaction model?