Data 401
Gradient Descent

Dennis Sun

October 12, 2016

1. Review
2. Gradient Descent
3. Stochastic Gradient Descent

1 Review

2 Gradient Descent

3 Stochastic Gradient Descent

Review of Linear Regression

- Linear regression chooses $\boldsymbol{\beta}$ to minimize

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Today

Efficient ways to fit linear regression to massive data sets where p is large.

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Numerical Optimization

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2. Move $\boldsymbol{\beta}$ in the direction that will decrease L the most to get a new guess $\boldsymbol{\beta}^{(1)}$.

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4. Repeat until it is no longer possible to decrease the value of L.

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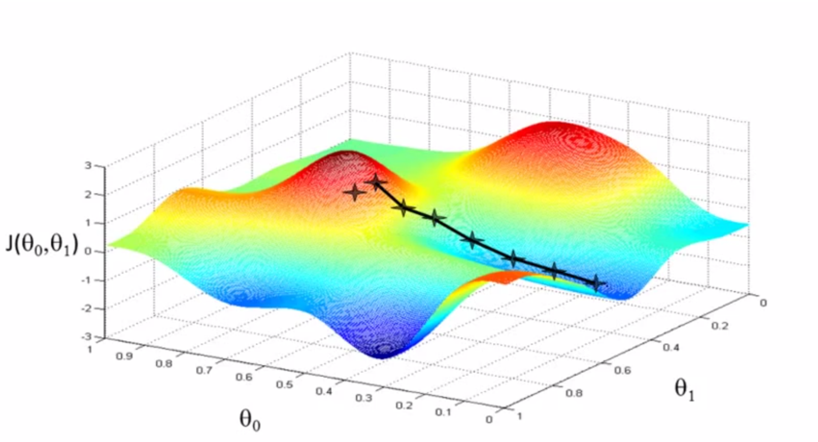

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What's the complexity of each iteration of gradient descent?
Answer: $O(np)$.

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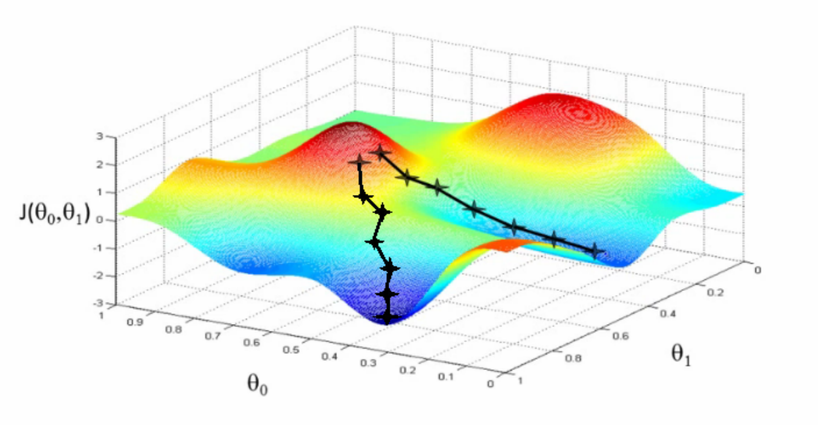
- How do we choose the learning rate?
- Different starting points can give us different answers.

Problems with Gradient Descent



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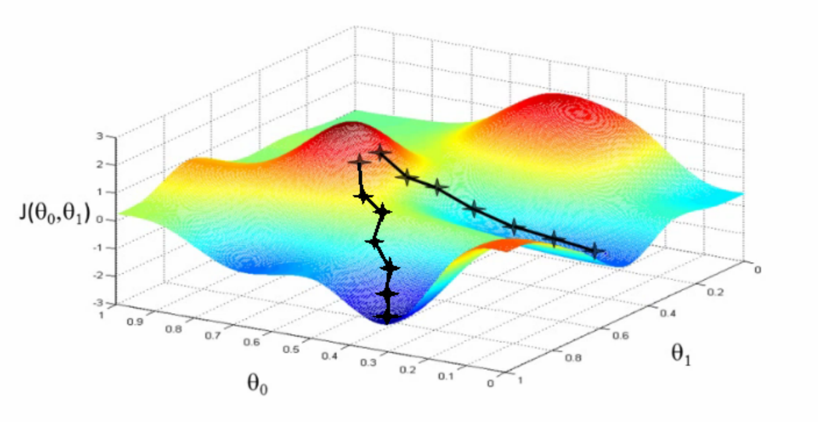

However, the objective function L for linear regression is **convex**, so it will only have different local minima. So gradient descent is guaranteed to converge to the minimizer (provided you choose α correctly).

The Gradient for Linear Regression

In-Class Exercise

The objective function for linear regression is

$$L(\boldsymbol{\beta}) = \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_{1i} + ... + \beta_p x_{pi}))^2.$$

Work out the gradient of L. Try to write your answer using linear algebra notation.

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$$\nabla L(\boldsymbol{\beta}) = -2X^T(\mathbf{y} - X\boldsymbol{\beta})$$

Implementing Gradient Descent for Linear Regression

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} - \alpha \nabla L(\boldsymbol{\beta}^{(k)})$$
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In-Class Exercise

Open the notebook `Gradient Descent for Linear Regression.ipynb`. *Implement the function* `lm_gd`.

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How do we update the coefficients $\boldsymbol{\beta}$? (**Note:** Updates need to be very efficient because the velocity of data may be high!)

Main Idea

Write the gradient as

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In other words, each observation has a contribution to the overall gradient.

Stochastic Gradient Descent

Move only in the direction of the gradient for the current observation:

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∇L_i will be much noisier, but much less computationally intensive to calculate. (It's only $O(p)$.)

Implementing Gradient Descent for Linear Regression

$$\boldsymbol{\beta}^{(i+1)} = \boldsymbol{\beta}^{(i)} - \alpha \nabla L_i(\boldsymbol{\beta}^{(i)})$$
$$\nabla L_i(\boldsymbol{\beta}) = -2(y_i - (\beta_0 + \beta_1 x_{i1} + ... + \beta_p x_{ip}))$$

In-Class Exercise

In the notebook `Gradient Descent for Linear Regression.ipynb`, *implement the function* `lm_sgd`.