Data 401
Maximum Likelihood

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1. The Class So Far
2. Probability Models and Maximum Likelihood
3. Linear Regression and Maximum Likelihood

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  - We will see how probability enters into linear regression.
- The unifying link will be the principle of **maximum likelihood**.

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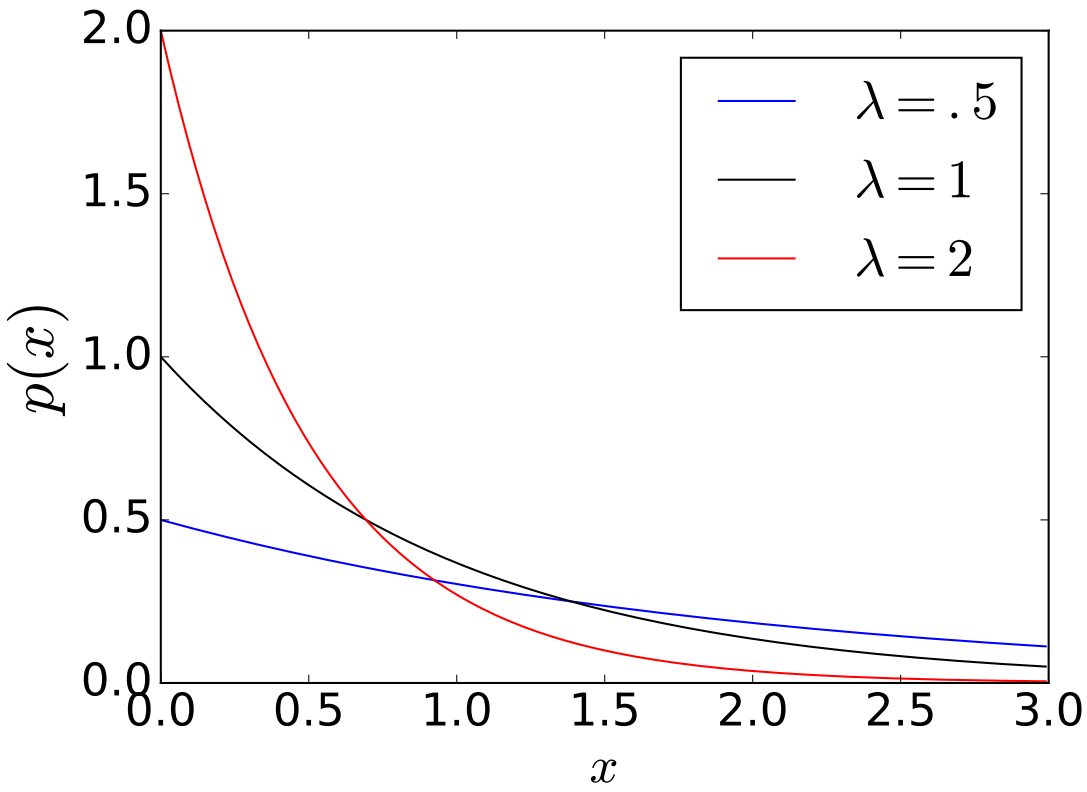
Suppose $X = 2\,\mathrm{years}$. How would you use this data to estimate λ?

Maximum Likelihood in Pictures

Idea: Find the λ which maximizes the probability of the data.



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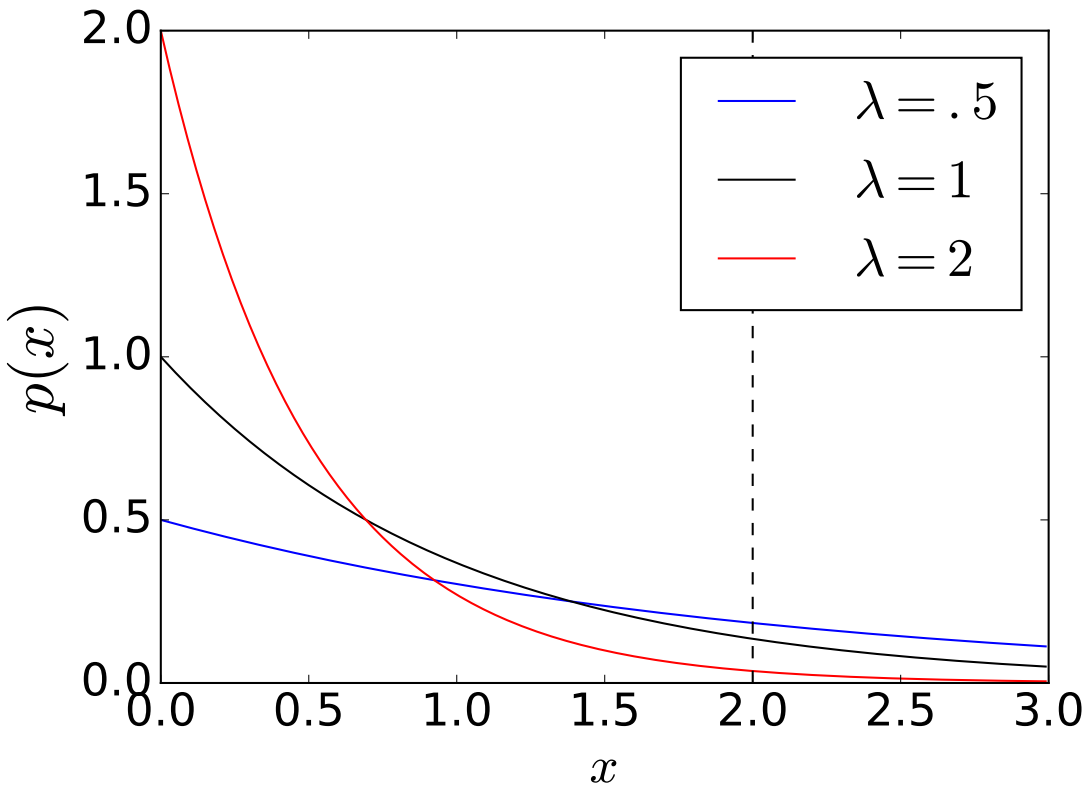

From this plot, it looks like $\lambda = .5$ is the best. But how do we know that there isn't some other value of λ that's even better?

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In other words, maximum likelihood requires that we view the p.d.f. as a function of λ, not as a function of x!

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L_X is called the **likelihood function**.

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So $\hat{\lambda}_{\text{MLE}} = \frac{1}{X}$. When $X = 2$, we get $\hat{\lambda}_{\text{MLE}} = .5$.

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Gradient descent / ascent!

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- Take the derivative (gradient) with respect to $\boldsymbol{\theta}$.
- If you can solve for $\hat{\boldsymbol{\theta}}$ by setting the gradient equal to $\mathbf{0}$, do it. Otherwise, use gradient ascent.

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The p.d.f. of a single Y_i is:

$$p_{\boldsymbol{\beta}}(Y_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(Y_i - (\beta_0 + \beta_1 X_{i1} + ... + \beta_p X_{ip}))^2},$$

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so the p.d.f. of all the Y_is is:

$$\begin{aligned} p_{\boldsymbol{\beta}}(Y_1, ..., Y_n) &= p_{\boldsymbol{\beta}}(Y_1) p_{\boldsymbol{\beta}}(Y_2) ... p_{\boldsymbol{\beta}}(Y_n) \\ &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(Y_i - (\beta_0 + \beta_1 X_{i1} + ... + \beta_p X_{ip}))^2}. \end{aligned}$$

Maximum Likelihood Estimator of β

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So the linear regression estimate is the MLE when we assume that the errors are normally distributed.