Data 401
Maximum Likelihood

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October 17, 2016
1. The Class So Far

2. Probability Models and Maximum Likelihood

3. Linear Regression and Maximum Likelihood
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3. Linear Regression and Maximum Likelihood
The Class So Far

• We've focused on linear regression, which
  minimizes the sum of squared differences,
  \[ \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i + ... + \beta_p x_{ip}))^2. \]

• Today's lecture, we will kill two birds with one stone:
  • We'll justify why we minimize the sum of squared differences, in
    place of some other metric.
  • We'll see how probability enters into linear regression.
  • The unify ink will be the principle of maximum likelihood.
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Probability Models

- Robability-ODELS
  - nSTaTISTICS,WEaSSUMETHaTOURDaTaCOMESFROMSOME
    - PROBaBILITYMODEL.
  - THISMODELCaPTURESEITHERTHERanDOMnESSInHEREnTInOUR
    - DaTaCOLLECTIOnPROCEDURE(E.G.,SIMPLERanDOMSaMPLE,
      - RanDOMIzEDExPERIMEnTS)ORnOISEInOURDaTa.
  - THEPROBaBILISTICMODELUSUaLLYSPECI
    - fi
      - ESTHEGEnERaL
        - FAMILyoFDISTRIbuTIoNS
          - BUTnOTTHEExaCTPaRaMETERS.
  - ExAMPLE: 3UPPOSEWEMIGHTMODELTHELIFETIMEOFaLIGHTBULB
    - aSanExPOnEnTIaLDISTRIBUTIoN,WITHP.D.F.
      - λ
        - (x) = λe^−λx.
Probability Models

• In statistics, we assume that our data comes from some probability model.

Example: Suppose we might model the lifetime of a light bulb as an exponential distribution, with P.D.F. 

\[ p_{\lambda}(x) = \lambda e^{-\lambda x} \]
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We still have to estimate $\lambda$.  

The Estimation Problem

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We have a lightbulb and we observe how long it lasts. That lifetime \( X \) is a random variable drawn from this distribution.
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Suppose \( X = 2 \) years. How would you use this data to estimate \( \lambda \)?
Maximum Likelihood in Pictures

Idea: Find the $\lambda$ which maximizes the probability of the data.
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From this plot, it looks like $\lambda = .5$ is the best. But how do we know that there isn’t some other value of $\lambda$ that’s even better?
Maximum Likelihood in Algebra

\[ p_\lambda(x) = \lambda e^{-\lambda x} \]

You are used to fixing a value of \( \lambda \), say \( \lambda = 1 \), and calculating how likely we are to observe a particular \( x \).
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Maximum likelihood involves inverting this perspective. We fix the value of \( x \) depending on the observed data and vary \( \lambda \).
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In other words, maximum likelihood requires that we view the p.d.f. as a function of \( \lambda \), not as a function of \( x \)!

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\( L_X \) is called the **likelihood function**.
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So $\hat{\lambda}_{\text{MLE}} = \frac{1}{X}$. When $X = 2$, we get $\hat{\lambda}_{\text{MLE}} = .5$. 
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What do we do for all the other cases where we can’t solve this equation?
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Gradient descent / ascent!
The Maximum Likelihood Recipe

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- Calculate \( \log p_{\theta}(X) \).
- Take the derivative (gradient) with respect to \( \theta \).
- If you can solve for \( \hat{\theta} \) by setting the gradient equal to 0, do it. Otherwise, use gradient ascent.
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The p.d.f. of a single $Y_i$ is:

$$p_\beta(Y_i) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} (Y_i-(\beta_0+\beta_1 X_{i1}+...+\beta_p X_{ip}))^2},$$
**Probability Model for Linear Regression**

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so the p.d.f. of all the $Y_i$s is:

$$p_\beta(Y_1, \ldots, Y_n) = p_\beta(Y_1)p_\beta(Y_2)\ldots p_\beta(Y_n)$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} (Y_i - (\beta_0 + \beta_1 X_{i1} + \ldots + \beta_p X_{ip}))^2}.$$
Maximum Likelihood Estimator of $\beta$

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$$= \sum_{i=1}^{n} \log \frac{1}{\sigma \sqrt{2\pi}} + \sum_{i=1}^{n} -\frac{1}{2\sigma^2} (Y_i - (\beta_0 + \beta_1 X_{i1} + \ldots + \beta_p X_{ip}))^2$$
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So the linear regression estimate is the linear regression estimate when we assume that the errors are normally distributed.
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