Name: ______________________________ Score: / 10

Data 451
A Formula for the Fibonacci Sequence

This assignment, should you choose to do it, is worth up to +1 extra credit point, applied to the PCA quiz. You may consult references, but you may not discuss the problems with anyone other than the instructors.

Description

The Fibonacci sequence is defined recursively, as follows. First, $x_0 = 1$ and $x_1 = 1$. Then, each successive number in the sequence is the sum of the two previous numbers, i.e., $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$.

Therefore, the first 10 Fibonacci numbers are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

It's easy to write code to calculate the nth number in the sequence. But is it possible to come up with an explicit formula for the nth Fibonacci number? This is what you will explore in this exercise.

Questions

Please type or write your answers on a separate sheet of paper.

1. (2 points) Consider the following matrix-vector expression that generates the Fibonacci sequence:

$$\begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{n-2} \\ x_{n-1} \end{bmatrix}.$$

   Let's call this matrix A. Suppose you know A^m for any power m. Explain how you could use this to calculate x_n, the nth Fibonacci number.

2. (5 points) Of course, calculating A^m directly is quite hard, much harder than just calculating the Fibonacci sequence. However, if we can find a basis of eigenvectors of A, then we can *diagonalize* the matrix A as

$$A = VDV^{-1},$$

   where V is a matrix whose columns are the eigenvectors and D is a diagonal matrix of the corresponding eigenvalues. Then, $A^m = VD^mV^{-1}$, and raising diagonal matrices to the mth power is easy (just raise each element to that power).

   Find the eigenvectors and eigenvalues of A. Form V and D, and check that $A = VDV^{-1}$. (You may need to review how to calculate eigenvectors and matrix inverses.)

3. (3 points) Use the diagonalization $A = VDV^{-1}$ you calculated above to work out A^m for any m. Then, use this to work out a formula for x_n, the nth Fibonacci number. The formula is quite surprising and involves the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.