Index Structures: Part 2
B+-trees

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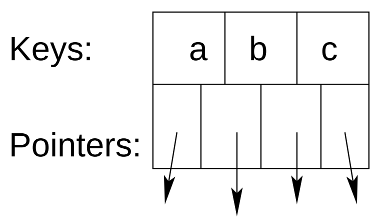
Just as simple index structures, *B+trees* are designed to index the content of existing database relations/data files in DBMS.

A *B+-tree* is a ballanced tree data structure defined as follows:

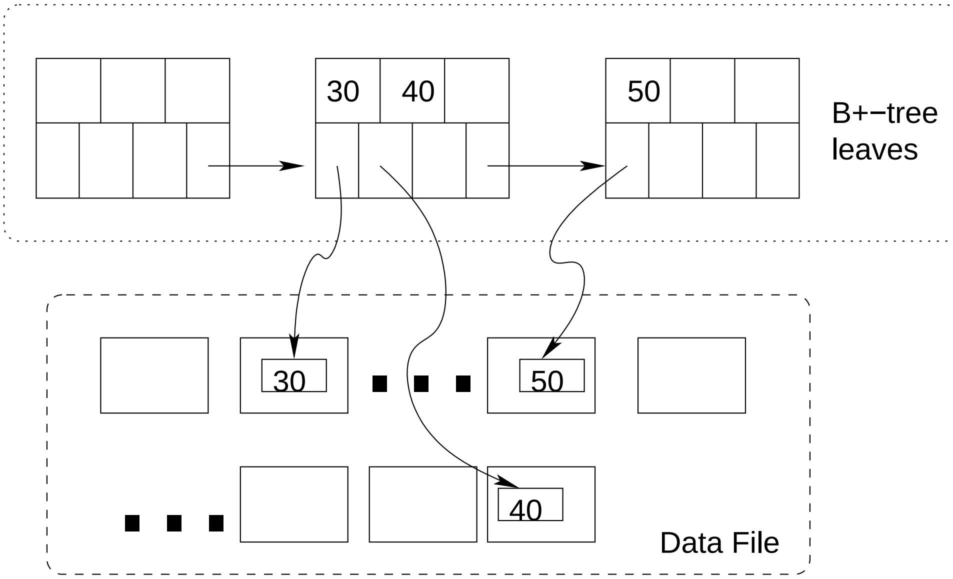
- Each node in a *B-tree* consists of n key values and $n + 1$ pointers. Figure below shows the node structure for $n = 3$. For simplicity, we will write that a node N is a pair $\langle Keys[0..n], Pointers[0..n + 1]\rangle$.

B+tree Node, n=3



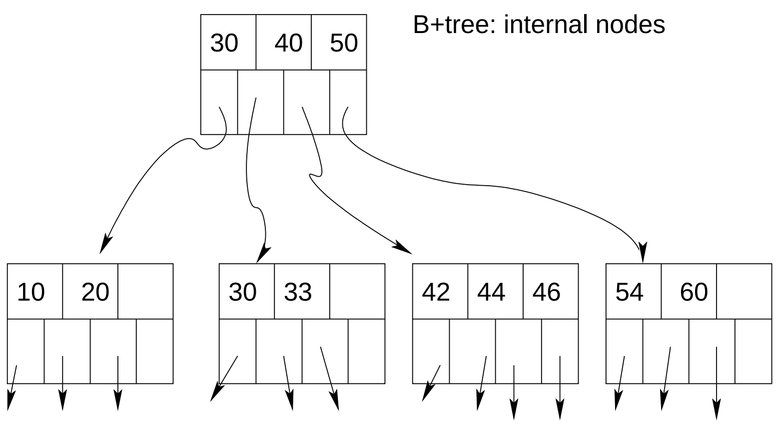
- *B+-trees* have three types of nodes: root, internal and leaf.
- Leaf nodes are constructed as follows:
  - At least half of the key value *slots* in each leaf node is *not empty*.
  - Given a leaf node $l = \langle Keys[], Pointers[]\rangle$, if $l.Keys[i] = a$ (not empty) then $l.Pointers[i]$ contains a pointer to a record with key value a. in the data file.
  - $l.Keys[i] <= l.Keys[i + 1]$ for all non-empty slots.
  - The last pointer of the leaf node, $l.Pointers[n + 1]$ points to the next leaf node l'. If k is the number of non-empty key values in l, then $l.Keys[k] <= l'.Keys[1]$.

The structure of leaf nodes is illustrated below.



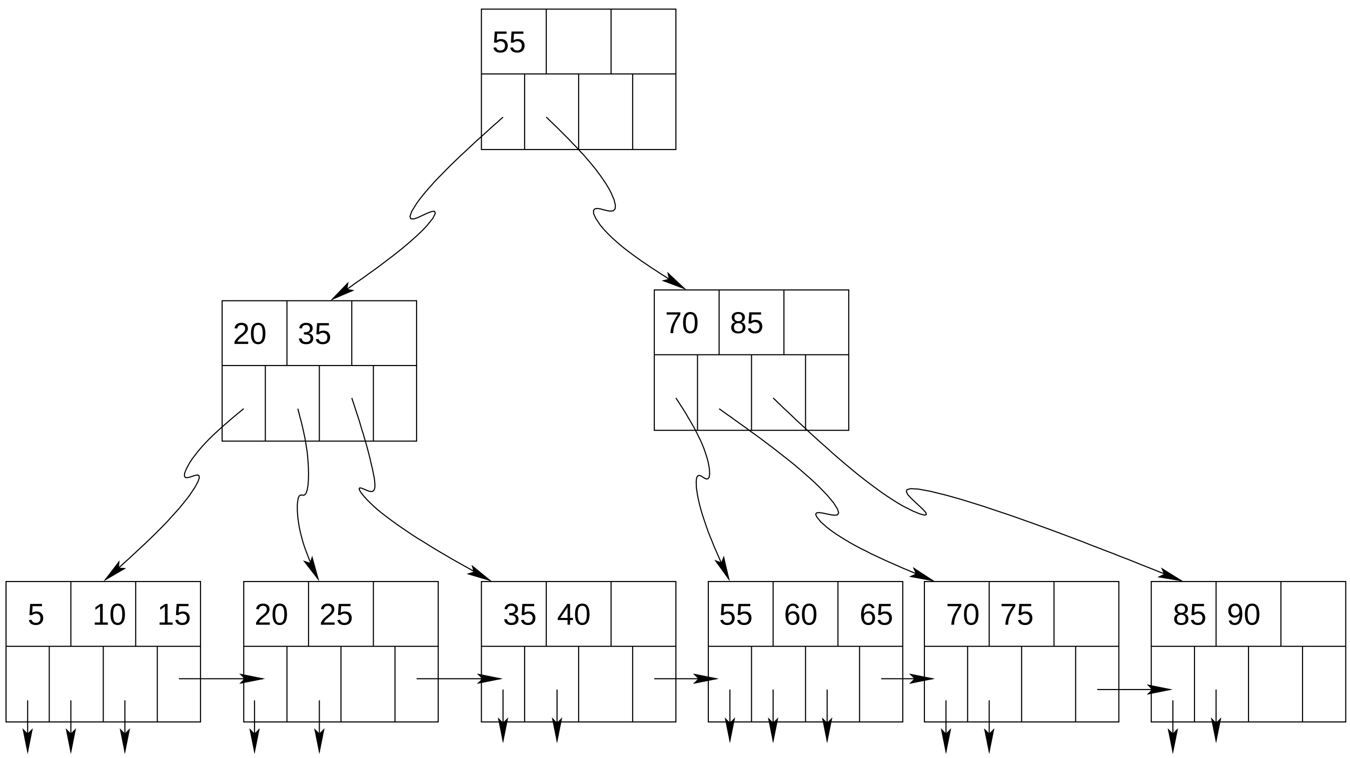
- Internal nodes have the following structure:
  - Each internal node has at least half of its key value slots occupied.
  - Given an internal node $N = \langle Keys, Pointers\rangle$, if $N.Keys[i] = a_i$ is non-empty, then $N.Pointers[i] \neq NULL$ and points to a node in the next level of the *B+-tree*. This node may be a *leaf* node, or another *internal* node.
  - If $N.Pointers[i] = N'$, then for $j \leq n$, if $N'.Keys[j]$ is not empty, then $N'.Keys[j] \leq N'.Keys[i]$.
    Additionally, if $i > 1$, $N'Keys[j] \geq N.Keys[i-1]$.
  - If $N.Keys[n] = a_n$ is not empty, then $N.Pointers[n+1] \neq NULL$ points to a node N' in the next level of the *B+-tree*, and all nonempty keys $N'.Keys[j] \geq N.Keys[n]$.

The structure of internal nodes is shown below:



- Root node. The structure of the root node of the *B+-tree* is similar to the structure of the internal node, with the exception that *root node may contain* *fewer than half of its key value slots occupied*. Instead, at least 2 pointers (and 1 key) in the root node must be non-empty.

A simple *B+-tree* for $n = 3$ is shown below:



B+-trees and Indexing Database Records

B-trees, and *B+-trees* are ballanced trees with a guarantee that beyond the root node, all other nodes are rather dense (i.e., filled at 50% or more).

The search algorithm over *B-trees* and *B+-trees* is straightforward:

- given a key value X, starting at the root node, traverse the key values stored in the node, until a value $Y > X$ is discovered at some slot i.
- Retrieve the node $Pointers[i]$.
- If all non-empty key values in the node are smaller than X, follow the last non-empty pointer.

B+-trees are an adaptation of the standard *B-tree* structure to the secondary storage. Each node of a *B+-tree* has the size of *one disk block*. The data portion of the disk block is broken into n $(Key, Pointer)$ pairs, and an additional, $n + 1$st pointer is stored at the end of the page.

The second distinction of *B+-trees* is the fact that all *leaf nodes* are linkes with each other. This makes it easy to search for keys in a sequence: searching for a starting position is done by traversing the tree, but after the first leaf node is retrieved, one can follow the $n+1$st pointer on the page, to retrieve the next leaf node.

Note: we also note that while the standard structure of a *B+-tree* assumes only a single-linked list of leaf nodes, we can also store a pointer to previous leaf node in the block header of each leaf node page.

B+-trees can be used to store any of the index structures discussed before:

- *Dense indexes* on sequential files. The leaf nodes form the dense index, while the upper layers provide fast navigation to the necessary key.
- *Sparse indexes* on sequential files. Same as above, leaf nodes form the sparse index.
- *Secondary indexes*. Leaf nodes present all key occurrences in sorted order.
- *Indexes with duplicate keys*. *B+-trees* need to be slightly updated to allow for seamless indexing of data with duplicate keys. In particular, the meaning of a key in an internal node has to change somewhat.

How many layers?

Suppose our disk blocks contain 4Kb each, 4096 bytes. Let our key values be integers, 4 bytes long and let our pointers be 8 bytes in size.

How many key values can we store in a single node?

We know that $12n + 8 + HeaderSize \leq 4096$. If we take $HeaderSize$ to be 80 bytes, this would lead to $12n = 4008$, or

$$n = 334.$$

A one-level $B+-tree$ (root and leaves) can thus index $334^2 = 111,556$ records. A two-level $B+-tree$ (root, internal layer and leaves) can index $334^3 = 37,259,704$ records.