Hybrid Probabilistic Programs

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Abstract

The precise probability of a compound event (e.g. $e_1 \vee e_2, e_1 \wedge e_2$) depends upon the known relationships (e.g. independence, mutual exclusion, ignorance of any relationship, etc.) between the primitive events that constitute the compound event. To date, most research on probabilistic logic programming [20, 24, 25, 26] has assumed that we are ignorant of the relationship between primitive events. Likewise, most research in AI (e.g. Bayesian approaches) has assumed that primitive events are independent. In this paper, we propose a *hybrid* probabilistic logic programming language in which the user can explicitly associate, with any given probabilistic strategy, a conjunction and disjunction operator, and then write programs using these operators. We describe the syntax of hybrid probabilistic programs, and develop a model theory and fixpoint theory for such programs. Last, but not least, we develop three alternative procedures to answer queries, each of which is guaranteed to be sound and complete.

1 Introduction

Though there has now been considerable work in the area of *quantitative logic programming* by many different authors [2, 13, 30, 36, 18], there has been relatively little work in the area of probabilistic logic programming [21, 20, 24, 25, 26]. The reason for this is that while connectives in multivalued logics can be interpreted in terms of the lattice's LUB (for disjunction) and GLB (for conjunction) operators, the same is not true in the case of probabilities. In particular, there is no single "formula" for computing the probability of a complex event $(e_1 \wedge e_2)$ where e_1, e_2 are primitive events. For instance:

1. If e_1, e_2 are *independent*, then $\mathbf{Prob}(e_1 \wedge e_2) = \mathbf{Prob}(e_1) \times \mathbf{Prob}(e_2)$.
2. If we are *ignorant* about the relationship between e_1, e_2, then all we can say[24] is that $\mathbf{Prob}(e_1 \wedge e_2)$ lies in the interval:

$$[\max(0, \mathbf{Prob}(e_1) + \mathbf{Prob}(e_2) - 1), \min(\mathbf{Prob}(e_1), \mathbf{Prob}(e_2))]$$

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This formula was first established by Boole[5] and forms the basis for many existing probabilistic logic treatments [12, 27, 24, 26]. [24] shows how these expressions are derived using a linear program, as does [37].

3. If we know that e_1, e_2 are *mutually exclusive*, then $\mathbf{Prob}(e_1 \wedge e_2) = 0$.
4. If we know that event e_1 implies event e_2 (called *positive correlation*), then $\mathbf{Prob}(e_1 \wedge e_2) = \mathbf{Prob}(e_1)$.

The above list represents a small fraction of relationships between events, each leading to different possible probabilities for complex events such as $(e_1 \wedge e_2)$. *The same holds for disjunctive events as well.*

In most previous efforts, probabilistic logic programming has assumed a fixed probabilistic strategy[21, 20, 24, 25, 26], such as (i) ignorance of the dependencies between events, or (ii) independence between events. (There are some exceptions to this, such as [33, 20]). However, an end user writing a probabilistic logic program should have the flexibility to write rules that reflect his/her specific knowledge about dependencies between events. For instance, the user should be able to express statements such as the two given below, that allow the user to explicitly articulate the probabilistic dependencies between events.

- "If the chairman of company C sells his stock and the chairman retires, *and we are ignorant of the dependencies between these two events*, then conclude that the stock in company C will drop, with probability between 40–90%."
- "If the chairman of company C sells his stock and the chairman retires, *and the retirement implies sale of stock* (e.g. in an employee owned company), then conclude that the stock in company C will drop, with probability between 5–20%."

Both rules above refer to the same two events, viz. sale of stock by the chairman, and retirement of the chairman. However, the first rule specifies what to conclude if we are ignorant of the relationship between these two events, while the second explicitly encodes specific knowledge about the dependencies between events. The rules lead to very different conclusions.

In this paper, we make the following contributions:

1. First, we define a general axiomatic notion of a *probabilistic strategy*. We show how a number of well known probabilistic strategies are special cases of our definition.
2. We then define the concept of a *hybrid probabilistic program* (*hp-program*, for short). If the user selects a set of probabilistic strategies $i_1, \ldots, i_k$ for use in an hp-program (s/he may select these in any way, as long as these selections satisfy the axioms defining probabilistic strategies), then this automatically defines a set of conjunction and disjunction connectives.
3. Subsequently, we define a fixpoint semantics for hp-programs, a model theoretic semantics for hp-programs, and a proof procedure, and prove that the fixpoint theory, model theory, and proof theory all lead to equivalent characterizations. *This applies to any selection of probabilistic strategies made by the user, as long as these selections satisfy the axioms defining probabilistic strategies.*
4. We then define a cache-based proof procedure that extends the well known work of Tamaki and Sato[34] to handle hybrid probabilistic programs. This procedure is also proved to be sound and complete.

2 Probabilistic Strategies (p-strategies)

In this section, we provide an axiomatic definition of probabilistic strategies (p-strategies). Intuitively, a p-strategy will specify different ways of computing probabilities of complex events, based on *knowledge* that the user may have about dependencies between the primitive events involved.

As we have already seen in the Introduction through the *ignorance* strategy, the probability of a compound event may be an *interval*, rather than a point, even if point probabilities are known for the primitive events involved. This was first shown by Boole[5] in 1854 and later used in [24, 26]. Both [24] and [37] describe the derivation of this expression by solving a linear program[1].

Thus, p-strategies will be defined on intervals – points, in any case, are special cases of intervals. Let $\mathcal{C}[0,1]$ denote the set of all closed intervals of $[0,1]$. There are two natural orderings on $\mathcal{C}[0,1]$.

- If $[a,b] \in \mathcal{C}[0,1], [c,d] \in \mathcal{C}[0,1]$ then we write $[a,b] \leq_t [c,d]$ if $a \leq c$ *and* $b \leq d$. According to this ordering on closed intervals, if an event e is assigned an interval $[a,b]$, and an event e' is assigned an interval $[c,d]$ such that $[a,b] \leq_t [c,d]$, then in fact it is more likely that event e' will occur, as its probability is "closer" to 1. In [21], Lakshmanan and Sadri introduced a similar ordering on *pairs* of intervals.
- Alternatively, we could use an inclusion ordering on intervals. Thus, if an event e is assigned an interval $[a,b]$, and an event e' is assigned an interval $[c,d]$ such that $[a,b] \subseteq [c,d]$, then our knowledge about event e is *more precise* than our knowledge about event e'.

Both these orderings will be used in this paper, for somewhat different purposes.

Throughout this paper, given a set X, we will use the notation 2^X to denote the power set of X. Thus, $2^{\mathcal{C}[0,1]}$ denotes the powerset of $\mathcal{C}[0,1]$. It is easy to see that 2^X is always a complete lattice under the ordering of inclusion.

A probabilistic strategy, defined below, is a pair of functions that satisfy certain axioms.

Definition 1 *A probabilistic strategy (p-strategy) is a pair of functions: $\rho = < c, d >$, such that:*

1. $c : \mathcal{C}[0,1] \times \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$ *is called a probabilistic composition function satisfying the following axioms:*
   (a) Commutativity : $c([a_1,b_1],[a_2,b_2]) = c([a_2,b_2],[a_1,b_1])$
   (b) Associativity : $c(c([a_1,b_1],[a_2,b_2]),[a_3,b_3]) = c([a_1,b_1],c([a_2,b_2],[a_3,b_3]))$
   (c) Inclusion-Monotonicity : $c([a_1,b_1],[a_2,b_2]) \subseteq c([a_3,b_3],[a_2,b_2])$ if $[a_1,b_1] \subseteq [a_3,b_3]$
   (d) Separation: *There exist two functions* $c^1, c^2 : [0,1] \times [0,1] \to [0,1]$ *such that* $c([a,b],[c,d]) = [c^1(a,c), c^2(b,d)]$.
2. $d : \mathcal{C}[0,1] \longrightarrow 2^{(\mathcal{C}[0,1] \times \mathcal{C}[0,1])}$ *is called a probabilistic decomposition function.*

A few comments on the axioms above are in order. The function c above is a *composition* function that generates a new interval from two input intervals. If the two input intervals denote the probabilities

[1]The basic intuition is this. Let $p_1, \ldots, p_n$ be some arbitrary, but fixed set of propositional symbols. Let $w_1, \ldots, w_k$ $(k = 2^n)$ be all subsets of $\{q_1, \ldots, q_n\}$. Each w_i denotes a possible world, or Herbrand interpretations. Suppose we know that formulas $F_1, \ldots, F_m$ constructed out of the above symbols have probabilities $p_1, \ldots, p_m$ respectively. Boole [5] argues that the world is certain, and it is our beliefs about the world that are uncertain. Therefore, if z_i denotes the probability that world w_i is in fact the true state of the real world, then for each F_i, we know that $\Sigma_{w_j \text{ satisfies } F_i} z_j = p_i$. If we denote this equality by Eq_i, then we have a set of constraints $Eq_1, \ldots, Eq_{2^n}$. To find a probability for a given formula F, we must minimize (to get a lower bound) and maximize (to get an upper bound) the expression $\Sigma_{w_i \text{ satisfies } F} z_i$. It is easily proved that the optimal value of the minimization may differ from the optimal value of the maximization, and hence, even if we know the precise probabilities of some basic events, we may not be able to provide a point probability for a conjunctive event.

of two different events, and if we know that the p-strategy used is $\rho =< c, d >$, then the new interval should represent the probability of a compound event. This explains the *commutativity* and *associativity* axioms, as $p(e_1 \wedge e_2) = p(e_2 \wedge e_1)$ and $p((e_1 \wedge e_2) \wedge e_3) = p(e_1 \wedge (e_2 \wedge e_3))$ (where e_1, e_2 and e_3 are some events).

To explain the axiom of *inclusion monotonicity* we shall recall that the smaller the probability interval is, the more precise information about the probability of an event we have. Based on this, the claim of the axiom of *inclusion monotonicity* is that the probability of a compound event is known more precisely when the probabilities of the simple events are known more precisely. Throughout the rest of the paper, when we speak of the axiom of *monotonicity*, we refer to this axiom.

Finally, the *separation* axiom states that the lower bound of the interval returned by any composition function must depend only on the lower bounds of the arguments of the function, and likewise, the upper bound of the resulting interval must depend only on the upper bounds of the arguments. This is a reasonable assumption, as our interval probabilities are intended to extend the point probabilities. As we know, if precise probabilities of two events are known, the probability of their combination depends only on these probabilities and on the relationship between the events, i.e., it is really a function of *two* arguments. We will write $c = [c^1, c^2]$ to express the fact that composition function c computes lower bounds according to function c^1 and upper bounds according to function c^2.

In the rest of the paper we will consider another property of composition functions: *continuity*.

Definition 2 *A composition function $c = [c^1, c^2]$ is called* continuous **iff** *both c^1 and c^2 are continuous in both their arguments. Similarly, a p-strategy $\rho = \langle c, d \rangle$ is continuous* **iff** *c is continuous.*

All the p-strategies considered in this paper will be continuous.

The *decomposition* function d takes an interval as input, and returns as output, a set of pairs of intervals. For now, there is no "connection" that ties c and d together: this will be made later through the concept of *coherence* (Definition 4). P-strategies are of two types, depending upon whether they satisfy certain extra axioms.

Definition 3 Conjunctive and Disjunctive p-strategies

- A *p-strategy $< c, d >$ is called* a conjunctive p-strategy if it satisfies the following axioms:
  1. Bottomline : *It is always the case that* $c([a_1, b_1], [a_2, b_2]) \leq_t [min(a_1, a_2), min(b_1, b_2)]$
  2. Identity : $c([a, b], [1, 1]) = [a, b]$
  3. Annihilator : $c([a, b], [0, 0]) = [0, 0]$
- A *p-strategy $< c, d >$ is called* a disjunctive p-strategy if c satisfies the following axioms:
  1. Bottomline : $[max(a_1, a_2), max(b_1, b_2)] \leq_t c([a_1, b_1], [a_2, b_2])$
  2. Identity : $c([a, b], [0, 0]) = [a, b]$
  3. Annihilator : $c([a, b], [1, 1]) = [1, 1]$

While we have already used the inclusion ordering on $\mathcal{C}[0,1]$ to define inclusion monotonicity, the *Bottomline* axiom uses the $\leq_t$ ordering. The *Bottomline* axiom establishes (in accordance with probability theory) that the probability of a conjunction of two events cannot exceed the probability of either of them and similarly that the probability of a disjunction of two events cannot be smaller than the probability of either of the events. The axioms of *Annihilator* and *Identity* deal with borderline cases (i.e. with conjunctions and disjunctions of an arbitrary event with an absolutely certain or impossible event).

Intuitively, a composition function determines, given the probability ranges of two events, the probability range of their (either and- or or-) composition. A decomposition function may be thought of as the inverse of composition: given the probability range of the result (and/or-composition of two events) it returns the set of all possible pairs of initial probabilistic ranges for the two events. To ensure that this holds we need the following definition:

Definition 4 A *p-strategy* $< c, d >$ *is called* coherent if

$$(\forall [a, b] \in \mathcal{C}[0, 1])(([a_1, b_1], [a_2, b_2]) \in d([a, b]) \ \textbf{ iff } \ c([a_1, b_1], [a_2, b_2]) = [a, b].$$

Throughout the rest of this paper, we will use the expression p-strategy to refer to coherent p-strategies, i.e. only coherent p-strategies will be considered. Before investigating the properties of p-strategies, we present some simple examples below.

2.1 Examples of P-strategies

In this section, we will present examples of various probabilistic assumptions that have been used extensively in reasoning with uncertainty. In particular, we show how the definition of a p-strategy is rich enough to capture these assumptions.

2.1.1 Independence

The strategy of independence may be described as the conjunctive p-strategy $inc =< c_{inc}, d_{inc} >$ and the disjunctive p-strategy $ind =< c_{ind}, d_{ind} >$, where:

- The conjunctive p-strategy $inc =< c_{inc}, d_{inc} >$ is given by:

$$\begin{aligned} c_{inc}([a_1, b_1], [a_2, b_2]) &= [a_1 a_2, b_1 b_2]. \\ d_{inc}(a, b) &= \{< [a_1, b_1], [a_2, b_2] > | (a_1 a_2 = a \ \textbf{ and } \ b_1 b_2 = b)\}. \end{aligned}$$

- The disjunctive p-strategy $ind =< c_{ind}, d_{ind} >$ is given by:

$$c_{ind}([a_1, b_1], [a_2, b_2]) = [a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2]$$

$$d_{ind}([a, b]) \text{ contains } < [a_1, b_1], [a_2, b_2] > \in \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$$

iff

$$a_1 + a_2 - a_1 a_2 = a \ \textbf{ and } \ b_1 + b_2 - b_1 b_2 = b$$

2.1.2 Ignorance

When nothing is known about the relationship between the events we are forced to use p-strategies that reflect *ignorance* [5, 12, 27, 24, 26, 19, 21]. $igc =< c_{igc}, d_{igc} >$ below is a *conjunctive* ignorance strategy, while $igd =< c_{igd}, d_{igd} >$ is a disjunctive ignorance strategy.

- Conjunctive ignorance p-strategy
  $igc =< c_{igc}, d_{igc} >$, where

$$c_{igc}([a_1, b_1], [a_2, b_2]) = [\max(0, a_1 + a_2 - 1), \min(b_1, b_2)]$$

$$d_{igc}([a, b]) \text{ contains } < [a_1, b_1], [a_2, b_2] >$$

iff

$$\textbf{if } a = 0 \textbf{ then } a_1 + a_2 \leq 1$$

$$\textbf{if } a > 0 \textbf{ then } a_1 + a_2 - 1 = a$$

$$(b = b_1 \textbf{ and } b_2 \geq b_1) \textbf{ or } (b = b_2 \textbf{ and } b_1 \geq b_2)$$

- Disjunctive ignorance p-strategy
  $igd =< c_{igd}, d_{igd} >$, where

$$c_{igd}([a_1, b_1], [a_2, b_2]) = [\max(a_1, a_2), \min(1, b_1 + b_2)]$$

$$d_{igd}([a, b]) \text{ contains } < [a_1, b_1], [a_2, b_2] >$$

  iff

$$(a = a_1 \textbf{ and } a_2 \leq a_1) \textbf{ or } (a = a_2 \textbf{ and } a_1 \leq a_2)$$

$$\textbf{if } b = 1 \textbf{ then } b_1 + b_2 \geq 1$$

$$\textbf{if } b < 1 \textbf{ then } b_1 + b_2 = b$$

Sometimes we will use ig instead of igc or igd whenever it is clear from the context whether a conjunctive or disjunctive strategy is under consideration.

2.1.3 Positive Correlation

Sometimes we know that the fact that event e_1 has happened implies that some event e_2 also had to happen (e.g., one would assume that *"Jon rides a bus"* would imply *"Jon bought a ticket"*). Below are conjunctive and disjunctive strategies for this case.

- Conjunctive p-strategy
  $pcc =< c_{pcc}, d_{pcc} >$, where

$$c_{pcc}([a_1, b_1], [a_2, b_2]) = [\min(a_1, a_2), \min(b_1, b_2)]$$

$$d_{pcc}([a, b]) = \{< [a_1, b_1], [a_2, b_2] >\}$$

  iff

$$(a = a_1 \textbf{ and } a_2 \geq a_1) \textbf{ or } (a = a_2 \textbf{ and } a_1 \geq a_2)$$

  and

$$(b = b_1 \textbf{ and } b_2 \geq b_1) \textbf{ or } (b = b_2 \textbf{ and } b_1 \geq b_2)$$

- Disjunctive p-strategy
  $pcd =< c_{pcd}, d_{pcd} >$, where

$$c_{pcc}([a_1, b_1], [a_2, b_2]) = [\max(a_1, a_2), \max(b_1, b_2)]$$

$$d_{pcc}([a, b]) = \{< [a_1, b_1], [a_2, b_2] >\}$$

  iff

$$(a = a_1 \textbf{ and } a_2 \leq a_1) \textbf{ or } (a = a_2 \textbf{ and } a_1 \leq a_2)$$

  and

$$(b = b_1 \textbf{ and } b_2 \leq b_1) \textbf{ or } (b = b_2 \textbf{ and } b_1 \leq b_2)$$

2.1.4 Negative correlation

Sometimes, the fact that event e_1 took place means that event e_2 could not possibly happen. e.g., if "Jon came by bus" did happen, then "Jon came by train" did not. Here, we know that both events could not possibly happen together, and therefore there is no conjunctive p-strategy for negative correlation. However, it does make sense to ask about the probability that one of the events took place. The disjunctive p-strategy below does precisely that.

$$ncd =< c_{ncd}, d_{ncd} >,$$

, where

$$c_{ncd}([a_1, b_1], [a_2, b_2]) = [\min(1, a_1 + a_2), \min(1, b_1 + b_2)]$$

$$d_{ncd}([a, b]) = \{< [a_1, b_1], [a_2, b_2] >\}$$

such that:

$$\textbf{if } a = 1 \textbf{ then } a_1 + a_2 \geq 1$$

$$\textbf{if } a < 1 \textbf{ then } a_1 + a_2 = a$$

$$\textbf{if } b = 1 \textbf{ then } b_1 + b_2 \geq 1$$

$$\textbf{if } b < 1 \textbf{ then } b_1 + b_2 = a$$

2.2 Validity of Examples

The following result, which is immediately verifiable from the definitions, asserts that the seven p-strategies described here are all coherent.

Proposition 1 *inc, igc and pcc are* continuous conjunctive coherent p-strategies. Similarly, *ind*, *igd*, *pcd* and *ncd* are continuous disjunctive coherent p-strategies.

The proof of this proposition can be found in Appendix A.

2.3 Properties of P-Strategies

In this section, we define various aspects of p-strategies that will play a key role in the definition of our fixpoint semantics and our model theory. First, we need the following very simple property.

Claim 1 *Let* $\rho = \langle c, d \rangle$ *be a coherent p-strategy. Then a pair* $\langle [a_1, b_1], [a_2, b_2] \rangle \in d([a, b])$ **iff** $\langle [a_2, b_2], [a_1, b_1] \rangle \in d([a, b])$

Proof. By *commutativity* of composition functions, if $c([a_1, b_1], [a_2, b_2]) = [a, b]$ then $c([a_2, b_2], [a_1, b_1]) = [a, b]$. As ρ is a *coherent* p-strategy, both $\langle [a_1, b_1], [a_2, b_2] \rangle$ and $\langle [a_2, b_2], [a_1, b_1] \rangle$ are in $d[a, b]$. □

The simple claim above assures us that if $\langle [a_1, b_1], [a_2, b_2] \rangle \in d([a, b])$, then so is $\langle [a_2, b_2], [a_1, b_1] \rangle$.

Claim 2 *Let* $\rho = \langle c_\rho, d_\rho \rangle$ *be a coherent disjunctive or conjunctive p-strategy. Then:*

1. $c_\rho([0, 1], [0, 1]) = [0, 1]$.
2. *More generally,* $(\forall x, y \in [0, 1])(\exists z \in [0, 1])(c_\rho([x, 1], [y, 1]) = [z, 1]$ *and* $(\forall x, y \in [0, 1])(\exists z \in [0, 1])c_\rho([0, x], [0, y]) = [0, z])$.

Proof.

- c_ρ is a *conjunctive* p-strategy.
  1. $(\forall x, y \in [0, 1])(\exists z \in [0, 1])(c_\rho([x, 1], [y, 1]) = [z, 1])$.
     We know that by the axiom of *Identity* for conjunctive p-strategies, $c_\rho([x, 1], [1, 1]) = [x, 1]$. Since $y \in [0, 1], [1, 1] \subseteq [y, 1]$. Therefore, by the axiom of *monotonicity* we get $c_\rho([x, 1], [1, 1]) = [x, 1] \subseteq c_\rho([x, 1], [y, 1]) \subseteq [0, 1]$.
     From this it is clear that the upper bound of the interval for $c_\rho([x, 1], [y, 1])$ will be 1, which means that $c_\rho([x, 1], [y, 1]) = [z, 1])$ for some $z \in [0, 1]$.
  2. $(\forall x, y \in [0, 1])(\exists z \in [0, 1])(c_\rho([0, x], [0, y]) = [0, z])$.
     We know that be the axiom of *Annihilator* for conjunctive p-strategies, $c_\rho([0, x], [0, 0]) = [0, 0]$. Since $y \in [0, 1], [0, 0] \subseteq [0, y]$. Therefore, by the axiom of *monotonicity* we get $c_\rho([0, x], [0, 0]) = [0, 0] \subseteq c_\rho([0, x], [0, y]) \subseteq [0, 1]$.

From this it is clear that the lower bound of the interval for $c_\rho([0,x],[0,y])$ will be 0, which means that $c_\rho([0,x],[0,y]) = [0,z])$ for some $z \in [0,1]$.

- c_ρ is a *disjunctive* p-strategy.
  1. $(\forall x, y \in [0,1])(\exists z \in [0,1])(c_\rho([x,1],[y,1]) = [z,1])$.
     We know that be the axiom of *Annihilator* for disjunctive p-strategies, $c_\rho([x,1],[1,1]) = [1,1]$. Since $y \in [0,1]$, $[1,1] \subseteq [y,1]$. Therefore, by the axiom of *monotonicity* we get $c_\rho([x,1],[1,1]) = [1,1] \subseteq c_\rho([x,1],[y,1]) \subseteq [0,1]$.
     From this it is clear that the upper bound of the interval for $c_\rho([x,1],[y,1])$ will be 1, which means that $c_\rho([x,1],[y,1]) = [z,1])$ for some $z \in [0,1]$.
  2. $(\forall x, y \in [0,1])(\exists z \in [0,1])(c_\rho([0,x],[0,y]) = [0,z])$.
     We know that be the axiom of *Identity* for disjunctive p-strategies, $c_\rho([0,x],[0,0]) = [0,x]$. Since $y \in [0,1]$, $[0,0] \subseteq [0,y]$. Therefore, by the axiom of *monotonicity* we get $c_\rho([0,x],[0,0]) = [0,x] \subseteq c_\rho([0,x],[0,y]) \subseteq [0,1]$.
     From this it is clear that the lower bound of the interval for $c_\rho([0,x],[0,y])$ will be 0, which means that $c_\rho([0,x],[0,y]) = [0,z])$ for some $z \in [0,1]$. □

Given a pair $[a,b]$, the projection set of decomposition function d w.r.t. $[a,b]$ is the set of all $[a',b']$'s such that $[a',b']$ can be composed with some $[a'',b'']$ via the composition function c to yield $[a,b]$.

Definition 5 *Let $\rho =< c, d >$. The "decomposition projection set" πD is defined to be:*

$$\pi D_\rho([a,b]) = \{[a',b'] \in \mathcal{C}[0,1] | (\exists [a'',b''] \in \mathcal{C}[0,1])(< [a',b'],[a'',b''] > \in d([a,b]))\}$$

Intuitively speaking, projection functions are used as follows: suppose we know that the probability of (say) some compound event $(e_1 \wedge e_2)$ lies in the interval $[a,b]$, when $\wedge$ is computed w.r.t. some conjunctive p-strategy $\rho =< c, d >$. In this case, $\pi D_\rho([a,b])$ specifies the set of all possible probability intervals for e_1 (and likewise for e_2) that could have led to $(e_1 \wedge e_2)$'s probability interval being $[a,b]$. In other words, in order for $(e_1 \wedge e_2)$'s probability interval to be $[a,b]$, e_1's probability interval must have been an element in $\pi D_\rho([a,b])$, but we do not know which one.

As a consequence, e_1's probability may be as low as the smallest point in $\bigcup_{[x,y]\in \pi D_\rho([a,b])}[x,y]$, or as large as the largest member of $\bigcup_{[x,y]\in \pi D_\rho([a,b])}[x,y]$. This yields an interval for e_1's probability, and motivates the following definition.

Definition 6 *Let $\rho =< c, d >$ be a p-strategy. A "maximal interval" md for $d([a,b])$ is defined as*

$$md_\rho([a,b]) = [\min_{[a',b']\in \pi D_\rho([a,b])}(a'), \max_{[a',b']\in \pi D_\rho([a,b])}(b')].$$

When computing probabilities of primitive events from known probabilities of more complex events, we need to be able to compute "maximal intervals" efficiently. The following theorem gives us a constant time method to compute "maximal intervals" w.r.t. conjunctive and disjunctive p-strategies.

Theorem 1 *Suppose $\rho = \langle c, d\rangle$ is any* conjunctive coherent p-strategy *and $\rho' = \langle c', d'\rangle$ is any* disjunctive coherent p-strategy. *Then:*

1. $(\forall [a,b] \in \mathcal{C}[0,1])(md_\rho([a,b]) = [a,1])$.
2. $(\forall [a,b] \in \mathcal{C}[0,1])(md_{\rho'}([a,b]) = [0,b])$.

Proof.

1. Let $md_\rho([a,b]) = [a',b']$. Since ρ is conjunctive strategy, $c_\rho([a,b],[1,1]) = [a,b]$ (Identity), and since ρ is coherent, $[1,1] \in \pi D_\rho([a,b])$. Since $b' = \max_{[\hat{a},\hat{b}]\in \pi D_\rho([a,b])}(\hat{b})$, and $[1,1] \in \pi D_\rho([a,b])$, $b' = 1$.
   Since $c_\rho([a,b],[1,1]) = [a,b]$ and ρ is coherent, $[a,b] \in \pi D_\rho([a,b])$. By the bottomline axiom, $(\forall [\hat{a},\hat{b}] \in \pi D_\rho([a,b]))(a \leq \hat{a})$. Since $[a,b] \in \pi D_\rho([a,b])$ $a = \min_{[\hat{a},\hat{b}]\in \pi D_\rho([a,b])}(\hat{a})$, and therefore,

$a' = a$.

2. Let $md_{\rho'}([a,b]) = [a',b']$. Since ρ' is disjunctive strategy, $c_{\rho'}([a,b],[0,0]) = [a,b]$ (Identity), and since ρ' is coherent, $[0,0] \in \pi D_{\rho'}([a,b])$.. Therefore, since $a' = \min_{[\hat{a},\hat{b}]\in \pi D_{\rho'}([a,b])}(\hat{a})$, and $[0,0] \in \pi D_{\rho'}([a,b])$, $a' = 0$.
   Since $c_{\rho'}([a,b],[0,0]) = [a,b]$ and ρ' is coherent, $[a,b] \in \pi D_{\rho'}([a,b])$. By the bottomline axiom, $(\forall [\hat{a},\hat{b}] \in \pi D_{\rho'}([a,b]))(b \geq \hat{b})$. Since $[a,b] \in \pi D_{\rho'}([a,b])$ $b = \max_{[\hat{a},\hat{b}]\in \pi D_{\rho'}([a,b])}(\hat{b})$, and therefore, $b' = b$. □

2.4 Other p-strategies

A natural question that the reader may ask is what p-strategies exist, in addition to those that we have presented above. We present a couple of other example p-strategies below that are hybrids of the ones presented earlier, and then we discuss how other p-strategies may be constructed. For the sake of simplicity, we only discuss composition functions as decomposition functions can be derived from composition functions using the definition of coherence.

Example 1 (Mixed-Ignorance-Independence Strategies) Consider a situation where a user considers two events e_1, e_2 whose probabilities are known to be in the ranges $[a_1,b_1], [a_2,b_2]$ respectively. The user in question is not sure if e_1, e_2 are independent, but thinks they might be. As a consequence, he wants the resulting range to be obtained by tightening the range that the ignorance strategy would have returned, by taking his feeling that events e_1, e_2 may be independent into account. He could do this in many ways.

Pessimistic Mixed Strategy: The user may define a pessimistic conjunction strategy c_{pes} such that $c_{pes}([a_1,b_1],[a_2,b_2])$ is computed as follows.

1. Compute $[a,b] = c_{ind}([a_1,b_1],[a_2,b_2])$ and $[a',b'] = c_{igc}([a_1,b_1],[a_2,b_2])$.
2. Let $c_{pes}([a_1,b_1],[a_2,b_2]) = [\min(a,a'), \min(b,b')]$.

This function can be verified to be a conjunctive p-strategy by modifying the proof of Proposition 1 appropriately. In fact, $c_{pes}([a_1,b_1],[a_2,b_2])$ can be directly computed to be $[\max(0, a_1 + a_2 - 1), b_1 b_2]$.

The intuition is that the user is not sure whether the events e_1, e_2 whose conjunction is being considered are independent or not. He chooses to be cautious, and decides to use the smallest values returned for the lower and upper bounds. This approach may be justified in applications where we need to be biased towards assuming lower probabilities of events.

Optimistic Mixed Strategy: On the other hand, the user may tend to assume higher probabilities for complex events, e.g. in the cases of failures between components of a physical system where independence is suspected, but not known, a user a may choose to believe higher probabilities of failure. In a sense, the user is optimistic that the probability that the complex event happens is larger than the pessimistic approach above might suggest. Here, he may use the following optimistic conjunction strategy $c_{opt}([a_1,b_1],[a_2,b_2])$:

1. Compute $[a,b] = c_{ind}([a_1,b_1],[a_2,b_2])$ and $[a',b'] = c_{igc}([a_1,b_1],[a_2,b_2])$.
2. Let $c_{opt}([a_1,b_1],[a_2,b_2]) = [\max(a,a'), \max(b,b')]$.

Here too, $c_{opt}([a_1,b_1],[a_2,b_2])$ can be directly computed to be $[a_1 a_2, \min(b_1,b_2)]$.

Example 2 (Generalized Mixed Strategies) The reader may have already noted that the "code" given above to merge independence and ignorance according to pessimistic or optimistic approaches can be generalized to merge arbitrary p-strategies, both for conjunction and disjunction. For example, suppose we have two conjunctive p-strategies ρ_1, ρ_2. If

$$\begin{aligned} c_{\rho_1}([a_1,b_1],[a_2,b_2]) &= [a,b] \\ c_{\rho_2}([a_1,b_1],[a_2,b_2]) &= [a',b'] \end{aligned}$$

then we may define a pessimistic mix, $c_{pes}^{\rho_1,\rho_2}$, and an optimistic mix, $c_{opt}^{\rho_1,\rho_2}$, as follows:

$$\begin{aligned} c_{pes}^{\rho_1,\rho_2}([a_1,b_1],[a_2,b_2]) &= [\min(a,a'),\min(b,b')]. \\ c_{opt}^{\rho_1,\rho_2}([a_1,b_1],[a_2,b_2]) &= [\max(a,a'),\max(b,b')]. \end{aligned}$$

In fact, it is easy to verify that for all $[a_1,b_1],[a_2,b_2]$, $c_{pes}^{\rho_1,\rho_2}([a_1,b_1],[a_2,b_2]) \leq_t c_{opt}^{\rho_1,\rho_2}([a_1,b_1],[a_2,b_2])$. Thus, the pessimistic mix of two p-strategies always tends to produce lower probabilities than the optimistic mix, thus justifying their names.

In addition to the above *mixing* strategies that allow us to define various new p-strategies, we provide below, some general guidance on how yet other p-strategies may be constructed.

Suppose χ is an associative and commutative function (of which there are many!) which takes two intervals $[a_1,b_1],[a_2,b_2]$ as input, and produces an output interval $[a,b]$. For χ to be the composition part of a conjunctive p-strategy. χ must satisfy the Bottom Line and Inclusion Monotonicity axioms as well as the Annihilator and Identity axioms. Out of these four, Bottom Line and Inclusion Monotonicity jointly impose very strong restrictions on which χ's can be used in p-strategies.

Suppose $[a_1,b_1] \subseteq [a_3,b_3]$, and $[a_2,b_2]$ is any subinterval of $[0,1]$. Let $[a,b] = \chi([a_1,b_1],[a_2,b_2])$ and $[a',b'] = \chi([a_3,b_3],[a_2,b_2])$. It is easy to see that:

$$\begin{aligned} a_3 &\leq a_1 \\ b_1 &\leq b_3 \\ a' &\leq a \\ b &\leq b' \\ a &\leq a_1 \\ b &\leq b_1 \\ a' &\leq a_3 \\ b' &\leq b_3 \end{aligned}$$

The first two conditions follow as $[a_1,b_1] \subseteq [a_3,b_3]$. The third and fourth inequalities follow because Inclusion Monotonicity tells us that $[a,b] \subseteq [a',b']$. The last four inequalities follow from the Bottom Line axiom which tells us that $[a,b] \leq_t [a_1,b_1]$ and $[a',b'] \leq_t [a_3,b_3]$. Figure 1 shows the relationships between these values diagrammatically. An edge from x to x' means x is less than or equal to x'.

In addition, the axiom of identity says that when $[a_2,b_2] = [1,1]$ then $\chi([a_1,b_1],[a_2,b_2]) = [a_1,b_1]$ and $\chi([a_3,b_3],[a_2,b_2]) = [a_3,b_3]$. Diagrammatically, this means that the top row and the bottom row in Figure 1 must coincide when $[a_2,b_2] = [1,1]$. Similarly, when $[a_2,b_2] = [0,0]$, the bottom row collapses to one point, viz. 0 because a', a, b, b' are all set to 0 by the Annihilator Axiom.

Thus, to define a function χ in such a way that it is the composition function of a p-strategy, the reader must ensure that the diagram associated with χ looks like that shown in Figure 1, and exhibits the extremal properties mentioned in the previous paragraph when $[a_2,b_2] = [1,1]$ or $[a_2,b_2] = [0,0]$.

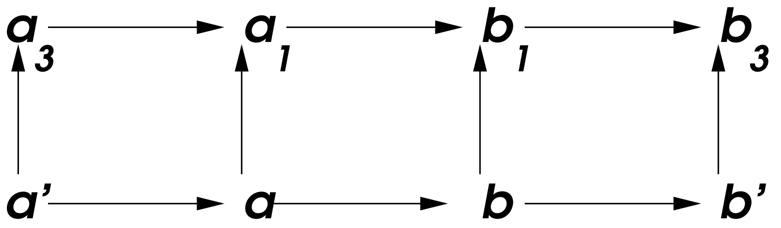

Figure 1: Inequalities induced by Bottomline and Inclusion Monotonicity Axioms

3 Syntax of hp-programs

In hybrid probabilistic programs, we assume the existence of an arbitrary, but fixed set of conjunctive and disjunctive p-strategies. The programmer may augment this set with new strategies when s/he needs new ones for their application. The following definition says that each conjunction strategy has an associated conjunction connective, and each disjunction strategy has an associated disjunction connective.

Definition 7 *Let $\mathcal{CONJ}$ be a finite set of conjunctive p-strategies and $\mathcal{DISJ}$ be a finite set of disjunctive p-strategies. Let $\mathcal{S}$ denote $\mathcal{CONJ} \cup \mathcal{DISJ}$.*

- Let $\rho \in \mathcal{CONJ}$. *Connective $\wedge_\rho$ is called an* ρ-annotated conjunction
- Let $\rho \in \mathcal{DISJ}$. *Connective $\vee_\rho$ is called an* ρ-annotated disjunction

Let L be a language generated by finitely many constant and predicate symbols. Let B_L denote the set of constant symbols (atoms) in L. We assume that L has no ordinary function symbols, but it may contain *annotation function* symbols for a fixed family of functions. The interpretation of these function symbols is given in definition 9 below.

Hybrid basic formulas, defined below, are either conjunctions of atoms, or disjunctions of atoms (but not mixes of both) w.r.t. a single connective.

Definition 8 *Let ρ be a* p-strategy*, and $A_1, \ldots, A_k$ be atoms. Then*

$$A_1 \wedge_\rho A_2 \ldots \wedge_\rho A_k$$

and

$$A_1 \vee_\rho A_2 \ldots \vee_\rho A_k$$

are called hybrid basic formulas*. Suppose $bf_\rho(B_L)$ denotes the set of all ground hybrid basic formulas for the $\vee_\rho$ and $\wedge_\rho$ connectives. Let $bf_{\mathcal{S}}(B_L) = \cup_{i \in \mathcal{S}} bf_\rho(B_L)$. Similarly, $bf_{\mathcal{CONJ}} = \cup_{i \in \mathcal{CONJ}} bf_\rho(B_L)$ and $bf_{\mathcal{DISJ}} = \cup_{i \in \mathcal{DISJ}} bf_\rho(B_L)$.*

For instance, returning to our stock example, the formulas (ch-sells-stock(C) $\vee_{igd}$ ch-retires(C)) and (price-drop(C) $\wedge_{inc}$ stable(C)) are basic formulas involving the ignorance and independence p-strategies. However, (price-drop(C) $\wedge_{inc}$ stable(C)) $\wedge_{ind}$ (price-drop(D) is not a basic formula, as it involves two different p-strategies. In order to proceed further we have to define a notion of *annotation*. Definitions 9 – 11 below were introduced in [25].

We are now ready to describe how annotation function symbols should be interpreted.

Definition 9 An *annotation function* f of arity n is a total function $f : [0,1]^n \longrightarrow [0,1]$.

Let $\mathcal{F}^n[0,1]$ denote an arbitrary, but fixed set of annotation functions of arity n and let $\mathcal{F}[0,1]$ denote $\cup_{n=0}^{\infty} \mathcal{F}^n[0,1]$.

We assume that associated with each *annotation function* is a body of software code computing that function, that is guaranteed to terminate on all inputs.

We also assume that all variable symbols from L are partitioned into two classes. We will call one class *object variable symbols* and this class will contain the regular first order logic variable symbols. The second class of variable symbols, *annotation variables* will contain variable symbols that range over the interval $[0, 1]$. These variables can appear only inside *annotation items*, which are defined below:

Definition 10 An *annotation item* δ is one of the following:

- *a constant in the* $[0, 1]$ *interval*
- *an annotation variable symbol from* L
- *let* f *be an annotation function symbol from* L *of arity* n *and let* $\delta_1, \ldots, \delta_n$ *be annotation items. Then* $f(\delta_1, \ldots, \delta_n)$ *is also an annotation item.*

Definition 11 *Let* δ_1 *and* δ_2 *be annotation items. Then* $[\delta_1, \delta_2]$ *is called an annotation or an annotation term.*

When δ_1, δ_2 are both constants, then the annotation term $[\delta_1, \delta_2]$ denotes an interval. Otherwise, it denotes a *set* of intervals, obtained by instantiating δ_1, δ_2 in different ways. Following the terminology introduced in [25] if an annotation term has no annotation variables in it, we call it a *c-annotation*. Otherwise it will be called a *v-annotation*.

Example 3 $[0, 1]$ *and* $[0.3, 0.6]$ *are* c-annotations*.* $[V_1, 1]$ *and* $[0.5 * V_1, V_1]$ *are* v-annotations*.*

Let B_L denote the Herbrand base of L. Since L contains no first-order logic function symbols, B_L is finite.

Definition 12 *A hybrid probabilistic annotated basic formula (hp-annotated basic formula) is an expression of the form* $B : \mu$ *where* B *is a hybrid basic formula and* μ *is an annotation .*

Informally speaking, $B : \mu$ may be read as "The probability of B occuring lies in the interval μ." For example, the annotated basic formula (ch-sells-stock(C) $\vee_{ig}$ ch-retires(C)):$[0.4, 0.9]$ may be read as: "The probability that the chairman sells stock or the chairman retires lies in the 40–90% interval, assuming (no knowledge) ignorance of the relationship between these two primitive events."

In this paper, hybrid probabilistic annotated basic formulas are the basic syntactic objects that *merge* together, probabilistic reasoning and logical reasoning. For example, if $(a \wedge_{ig} b) : [0.5, 0.7]$ is a hybrid probabilistic annotated basic formula, this formula says that "If we assume that we have no knowledge of the dependencies or lack thereof between events a and b, then the probability that both events a and b occur lies between 0.5 and 0.7 inclusive." In general, the hybrid probabilistic annotated basic formula $(a \wedge_\rho b) : [0.5, 0.7]$ says that "If we assume that our knowledge of the dependency between a and b is given by the probabilistic strategy ρ, then the probability that both events a and b occur lies between 0.5 and 0.7 inclusive." Similar rationales can be given for disjunctive basic formulas.

Hybrid rules may now be constructed from hybrid annotated formulas as follows.

Definition 13 *Let* $B_0, B_1, \ldots, B_k$ *be hybrid basic formulas. Let* $\mu_0, \mu_1, \ldots, \mu_k$ *be annotations, such that every annotation variable (if any) occurring in* μ_0 *also occurs in at least one of* $\mu_1, \ldots, \mu_k$ *. A hybrid probabilistic clause (hp-clause) is a construction of the form:*

$$B_0 : \mu_0 \leftarrow B_1 : \mu_1 \wedge \ldots \wedge B_k : \mu_k$$

Informally speaking, the above rule is read: "If the probability of B_1 falls in the interval μ_1 and $\cdots$ the probability of B_k falls within the interval μ_k, then the probability of B_0 falls within the interval μ_0." Intuitively a basic formula is a statement about probabilities of events. The conjunction in the body of an hp-clause, on the other hand defines a conjunction of such statements, but does not itself represent an event.

Notice that the definition above contains a requirement that every annotation variable that appears in the annotation for the head of the clause also appears in one or more annotations for the body of the hp-clause. Therefore,

Example 4
- $A : [V_1, V_1] \longleftarrow$ *is not an hp-clause.*
- $A : [V_1, V_2] \longleftarrow (B \wedge_{ind} C) : [0, V_1] \wedge D : [V_2, 1]$ *is an hp-clause.*

Definition 14 *A hybrid probabilistic program (hp-program) over set $\mathcal{S}$ of p-strategies is a finite set of hp-clauses involving only connectives from $\mathcal{S}$.*

For example, the following four clauses constitute a simple hp-program using the p-strategies of ignorance and independence.

STOCK PROGRAM

price-drop(C):$[0.4, 0.9] \longleftarrow$ (ch-sells-stock(C) $\vee_{igd}$ ch-retires(C)):$[0.6, 1]$.
price-drop(C):$[0.5, 1] \longleftarrow$ (strike(C) $\vee_{ind}$ accident(C)):$[0.3, 1]$.
buy-stock(C):$[0.7, 1] \longleftarrow$ (price-drop(C) $\wedge_{inc}$ stable(C)):$[0.3, 1]$.
sell-stock(C):$[0.5, 1] \longleftarrow$ (price-drop(C) $\wedge_{inc}$ unstable(C)):$[0.4, 1] \wedge$ have-stock(C):$[1, 1]$.
stable(c):$[0.8, 1] \longleftarrow$.
strike(c):$[0.4, 0.5] \longleftarrow$.
unstable(C):$[V1, V2] \longleftarrow$ stable(C):$[1 - V2, 1 - V1]$.

The above program is a very simple example of a market decision making program. The first two rules tell us when to expect that the stock of company C will drop. According to the first rule, it will drop with probability between 40% and 90% if the probability that the CEO of the company will sell the stock or that he will retire is more than 60%. We use the *ignorance* assumption here as we do not know if there is any connection between the two events. In fact, for different companies the *correlation* may range from the two being independent, to one being a consequence of the other. The *ignorance* assumption here gives us a "lowest common denominator" in terms of the relationship between the two events.

The second rule states that if the probability that the company's employees will go on strike or that an accident happens on the premises of the company is over 30%, then the probability that the stock of the company will drop is *at least* 50%. If we assume that causes of strikes and accidents are completely different, then the two events are independent of each other.

The next two rules deal with decision-making. The third rule of the program, says that we should buy stock of company C if its price drops and the company is generally known to be stable. We want to assume that our knowledge of the stability of company C is independent of the price drop under consideration, therefore, the conjunction of the two events is made under the assumption of independence. The fourth rule provides an alternative to the third by declaring that if the price drops and there is a high probability that the company is unstable, the stock has to be sold. For this rule to fire, however, we need one more condition: one can sell stock of company C only if one owns this stock. This is why we must know for sure (i.e. with probability 100%) that we own this stock if we want to sell it.

Two facts that follow describe our current knowledge of situation, expressed probabilistically. The first fact states that company c is stable with probability more than 80%. The second fact states that the probability of a strike for this company is between 40% and 50%.

Finally the last rule can be used to establish the connection between the information about the stability of company C and its nonstability. Indeed, if we assume that each company is either stable or unstable (a reasonable assumption for our example), then, if we know that the probability that company C is stable is p, the probability that C is unstable (i.e., *not stable*) than would have to be $1 - p$. We extend this simple observation to the notion of probabilistic intervals to obtain that if C is stable with probability between $V1$ and $V2$ then it is unstable with probability between $1 - V2$ and $1 - V1$. □

Example 5 *Let us consider a rule of the form*

$$c : \mu \quad \leftarrow \quad (a \wedge_{inc} b) : \mu_1 \wedge (a \wedge_{pcc} b) : \mu_2.$$

A rule of this sort may be read as "If $(a \wedge b)$'s probability lies in the interval μ_1 when a, b are assumed to be independent, and if $(a \wedge b)$'s probability lies in the interval μ_2 when a, b are assumed to be positively correlated, then c's probability lies in the interval μ." This rule contains no inconsistency as stated above. Rather, such a rule might reflect some doubt on the part of the author of the rule about the precise relationship between a and b — are they independent? Or are they positively correlated? □

If P is an hp-program, then we will write $ground(P)$ to represent the set of all ground instances of the rules from P.

4 Declarative Semantics of hp-programs

Having described the syntax of hp-programs, we are now in a position to develop the declarative semantics of such programs. We will first develop a fixpoint semantics of hp-programs, followed by a model theoretic semantics, and show that the two are essentially equivalent characterizations of hp-programs. Later, in Section 5, we will provide a proof procedure for hp-programs.

4.1 Fixpoint Semantics

As usual, suppose we have a logical language L consisting of variable symbols, constant symbols, function symbols, and predicate symbols, and let B_L denote the Herbrand base of this language. An atomic function, defined below, merely assigns closed intervals to ground atoms.

Definition 15 *A function $f : B_L \longrightarrow \mathcal{C}[0,1]$ is called an* atomic formula function *or* atomic function.

It is easy to see that the set of all atomic functions is a complete lattice. This is because if $(X, \sqsubseteq)$ is any complete lattice, then the set of all functions of the type $2^X \rightarrow 2^X$ is a complete lattice also under the pointwise ordering $f \sqsubseteq g$ iff $(\forall x \in X) f(x) \sqsubseteq g(x)$.

Though atomic functions do not, by themselves, make assignments to basic formulas, they may be extended to do so.

Before proceeding further we first introduce some notation for "splitting" a complex formula into two parts.

Definition 16 *Let $F = F_1 *_\rho \ldots *_\rho F_n$, $G = G_1 *_\rho \ldots *_\rho G_k$, $H = H_1 *_\rho \ldots *_\rho H_m$ where $* \in \{\wedge, \vee\}$. We write $G \oplus_\rho H = F$ (or $G \oplus H$ if the p-strategy ρ is irrelevant)* **iff:**

1. $\{G_1, \ldots, G_k\} \cup \{H_1, \ldots, H_m\} = \{F_1, \ldots, F_n\}$ *and*
2. $\{G_1, \ldots, G_k\} \cap \{H_1, \ldots, H_m\} = \emptyset$.
3. $k > 0$ *and* $m > 0$.

Definition 17 *A* hybrid formula function *is a function $h_f : bf_{\mathcal{S}}(B_L) \longrightarrow \mathcal{C}[0,1]$ which satisfies the following properties:*

1. Commutativity. *If $F = G_1 \oplus_\rho G_2$ then $h(F) = h(G_1 *_\rho G_2)$.*
2. Composition. *If $F = G_1 \oplus_\rho G_2$ then $h(F) \subseteq c_\rho(h(G_1), h(G_2))$.*
3. Decomposition. *For any basic formula F, $h(F) \subseteq md_\rho(h(F *_\rho G))$ for all $\rho \in \mathcal{S}$ and $G \in bf_{\mathcal{S}}(B_L)$.*

Given an atomic function f and a hybrid formula function *h we say that h is* based on *f* **iff** $(\forall A \in B_L)(f(A) = h(A))$. *Sometimes we will use notation h_f to represent the fact that h is based on f.*

From the first condition it follows that $h(F) = h(F')$ for any F and F' which are permutations of one another. The second condition states that the probability of a complex formula is bounded by the probabilities of its subformulas. Conversely, the third condition bounds the probability of a subformula by the probability of a formula it is a part of. Clearly, for each atomic function f, there exists an entire *family* of hybrid formula functions based on it. This corresponds to our intuition that the knowledge of the probabilities of atomic events does not necessarily allow us to uniquely compute the exact probabilities of complex events. In fact, it is possible to directly express the fact that we know the probability of a complex event. The only requirements we place on hybrid formula functions is that they provide **consistent** and **maximally tight** information about the probability intervals associated with both atomic and complex events.

As mentioned above, each atomic function f produces a family of hybrid formula functions h_f based on it. We let $h[f]$ denote the set of all hybrid formula functions based on atomic function f. Let $\mathcal{HFF}$ denote the set of all hybrid formula functions generated by some arbitrary but fixed set of p-strategies. The $\leq$-ordering on atomic functions may be extended to basic formulas in the obvious way: $h_1 \leq h_2$ iff $(\forall F \in bf_{\mathcal{S}}(B_L))\, h_1(F) \supseteq h_2(F)$.

Let f and g be two atomic functions and let $f \leq g$. One may wonder whether the statement $(\forall h \in h[f])(\forall h' \in h[g])(h \leq h')$ holds. The following example demonstrates that this statement is false.

Example 6 *Let $B_L = \{a, b\}$. We define the functions f and g as follows:*

$f(a) = [0.4, 0.8] \quad g(a) = [0.6, 0.6]$
$f(b) = [0.5, 0.7] \quad g(b) = [0.6, 0.6]$

Clearly $f \leq g$. Now we consider two hybrid formula functions $h \in h[f]$ and $h' \in h[g]$ defined on formula $a\wedge_{inc}$ as follows:

$h(a \wedge_{inc} b) = [0.4, 0.4]$
$h'(a \wedge_{inc} b) = [0.36, 0.36]$

We can see that both h and h' satisfy the Composition *and* Decomposition *properties of the hybrid formula functions:*

$h(a \wedge_{inc} b) = [0.4, 0.4] \subseteq c_{inc}(h(a), h(b)) = c_{inc}([0.4, 0.8], [0.5, 0.7]) = [0.2, 0.56]$.
$h(a) = [0.4, 0.8] \subseteq md_{inc}(h(a \wedge_{inc} b)) = md_{inc}([0.4, 0.4]) = [0.4, 1]$.
$h(b) = [0.5, 0.7] \subseteq md_{inc}(h(a \wedge_{inc} b)) = md_{inc}([0.4, 0.4]) = [0.4, 1]$.

$h'(a \wedge_{inc} b) = [0.36, 0.36] \subseteq c_{inc}(h(a), h(b)) = c_{inc}([0.6, 0.6], [0.6, 0.6]) = [0.36, 0.36]$.
$h'(a) = [0.6, 0.6] \subseteq md_{inc}(h(a \wedge_{inc} b)) = md_{inc}([0.36, 0.36]) = [0.36, 1]$.
$h(b) = [0.6, 0.6] \subseteq md_{inc}(h(a \wedge_{inc} b)) = md_{inc}([0.36, 0.36]) = [0.36, 1]$.

As $h(a \wedge_{inc} b) \not\subseteq h'(a \wedge_{inc} b)$ it is clear that $h \not\leq h'$.

In fact, the example above suggests, that even a weaker statement, $(\forall h \in h[f])(\exists h' \in h[g])(h \leq h')$, is false. To see this, observe that in this example $h[g] = \{h'\}$. The example above suggests that if there is a relationship between the orders of atomic and hybrid functions, this relationship is more subtle. The following theorem establishes this relationship.

Theorem 2 *Let f, g be two atomic functions and let $f \leq g$. Then*

$$(\exists h \in h[f])(\forall h' \in h[g])(h \leq h')$$

Proof. consider the function $h \in h[f]$ defined as follows: $h(F *_\rho G) = c_\rho(h(F), h(G))$. We will show that $(\forall h' \in h[g])(h \leq h')$.

Let us consider an arbitrary by fixed function $h' \in h[g]$. We prove that $h \leq h'$ by induction on the size of basic formula F. If F is an atom, then $h(F) = f(F) \leq g(F) = h'(F)$. Now, let $F = G *_\rho H$ and let $h(G) \leq h'(G)$ and $h(H) \leq h'(H)$. By definition of h, $h(F) = c_\rho(h(G), h(H))$. As we know that c_ρ is monotonic we get, $c_\rho(h(G), h(H)) \leq c_\rho(h'(G), h'(F))$. But since h' is a formula function, it satisfies Composition and Commutativity and hence $h'(F) = h'(G *_\rho H) \subset c_\rho(h'(G), h'(H))$ or $c_\rho(h'(G), h'(H) \leq h'(F)$. From the above inequalities we get: $h(F) = c_\rho(h(G), h(H)) \leq c_\rho(h'(G), h'(H)) \leq h'(F)$ which is the desired result. ■

In order to define the iterations of the T_P operator later in the paper, we need the following theorem.

Theorem 3 *Let $\mathcal{S}$ contain* only continuous p-strategies. *Let $H = h_1, h_2, \ldots$ be an infinite sequence of* fully defined *hybrid formula functions over $bf_{\mathcal{S}}(B_L)$, such that $h_i \leq h_{i+1}$ (we can call this* an ascending sequence*). H has a* **least upper bound**, *i.e., there exists such hybrid formula function h^* such that $(\forall i)(h_i \leq h^*)$ and for any other function h which is an upper bound of H, $h^* \leq h$.*

Proof.

Let F be some hybrid basic formula. If h is a formula function we will write $h(F) = [h^1(F), h^2(F)]$.

We know that the sequence $H_F^1 = h_1^1(F), h_2^1(F), \ldots$ is ascending and bounded (at least by 1). Therefore, by a well-known property of the sequences of real numbers, H_F^1 has a limit $x_F = \lim_{i\to\infty} h_i^1(F)$. By the same property, the descending sequence $H_F^2 = h_1^2(F), h_2^2(F), \ldots$ (bounded by 0) has a limit $y_F = \lim_{i\to\infty} h_i^2(F)$. Since all h_i are fully defined, $x_F \leq y_F$.

Now we define a function h^* as $h^*(F) = [\lim_{i\to\infty} h_i^1(F), \lim_{i\to\infty} h_i^2(F)]$. To prove the desired result it suffices to show that (i) h^* is an upper bound, (ii) h^* is the <u>least</u> upper bound and (iii) h^* is a valid hybrid formula function.

- <u>h^* is an upper bound of H.</u>
  We know that the limit of an ascending sequence is greater than or equal to any member of the sequence. Similarly, the limit of a descending sequence is less than or equal to any member of the sequence. But then, for any basic formula F, it is true that $(\forall i) h^*(F) = [\lim_{i\to\infty} h_i^1(F), \lim_{i\to\infty} h_i^2(F)] \subseteq [h_i^1(F), h_i^2(F)] = h_i(F)$. From this it directly follows that $(\forall i)(h_i \leq h^*)$, i.e., h^* is an upper bound of H.
- <u>h^* is the least upper bound of H.</u>
  Let h an upper bound of H, i.e., let $(\forall i)(h_i \leq h)$. Let F be some basic formula. Let us compare $h^1(F)$ and $h^{*1}(F)$. We know that $h^{*1}(F) = \lim_{i\to\infty} h_i^1(F)$. We also know that $(\forall i)(h_i^1(F) \leq h^1(F)$. Then, by the property of real sequences, $lim_{i\to\infty} h_i^1(F) \leq h^1(F)$. Similarly, we can establish that $h^2(F) \leq \lim_{i\to\infty} h_i^2(F)$. From these two inequalities we see that $h(F) \subseteq h^*(F)$, i.e., $h^* \leq h$.
- <u>h^* is a valid hybrid formula function.</u>
  To show this we have to prove that h^* satisfies the Commutativity, Composition and Decomposition.
  First we show that h^* satisfies the <u>Commutativity</u> property. Let F be some basic formula and let G and G' be basic formulas such that $F \equiv G \oplus_\rho G'$ for some p-strategy ρ. Since all functions h_i are valid hybrid formula functions, they all satisfy the Commutativity postulate, i.e., $(\forall i)(h_i(F) = h_i(G *_\rho G'))$, where $* \in \{\wedge, \vee\}$ depending on whether ρ is conjunctive or disjunctive. Then, the sequences $h_1(F), h_2(F), \ldots$ and $h_1(G *_\rho G'), h_2(G *_\rho G'), \ldots$ *coincide* and therefore the limits of the lower and upper bound sequences are the same as well, i.e. $\lim_{i\to\infty}(h_i^1(F)) = \lim_{i\to\infty}(h_i^1(G*_\rho G'))$ and $\lim_{i\to\infty}(h_i^2(F)) = \lim_{i\to\infty}(h_i^2(G*_\rho G'))$. However, since $h^*(F) = [\lim_{i\to\infty}(h_i^1(F)), \lim_{i\to\infty}(h_i^2(F))]$ and $h^*(G*_i G') = [\lim_{i\to\infty}(h_i^1(G*_\rho G')), \lim_{i\to\infty}(h_i^2(G*_\rho G'))]$, we get $h^*(F) = h^*(G *_\rho G')$.

Now we proceed to show that h^* satisfies the Composition postulate. Let F be some basic formula and let $F = G *_\rho G'$. We need to show that $h^*(F) \subseteq c_\rho(h^*(G), h^*(G'))$. Remember that c_ρ satisfies the axiom of Separation, i.e., $c_\rho = \langle c^1_\rho, c^2_\rho \rangle$.

First, we show that $c^1_\rho(h^{*1}(G), h^{*1}(G') \leq h^{*1}(F)$. By a similar argument we will then be able to establish that $h^{*2}(F) \leq c^1_\rho(h^*(G), h^*(G')$. The two statements will give us the desired result.

To show $c^1_\rho(h^{*1}(G), h^{*1}(G') \leq h^{*1}(F)$, we first note that $h^{*1}(F) = lim_{i \to \infty} h_i(F) \geq lim_{i \to \infty} c^1_\rho(h^1_i(G), h^1_i(G'))$. We know this because, since all h_i satisfy the Composition axiom – hence $(\forall i)(h^1_i(F) \geq c^1_\rho(h^1_i(G), h^1_i(G'))$, and therefore the sequence $h^1_1(F), h^1_2(F), \ldots$ *dominates* [2] the sequence $c^1_\rho(h^1_1(G), h^1_1(G')), c^1_\rho(h^1_2(G), h^1_2(G')), \ldots$. Then, the limit of the former sequence has to be greater than or equal to the limit of the latter.

As we know that c_ρ is a **continuous** p-strategy, c^1_ρ is continuous in both arguments. Therefore $\lim_{i \to \infty} c^1_\rho(h^1_i(G), h^1_i(G')) = c^1_\rho(\lim_{i \to \infty} h^1_i(G), \lim_{i \to \infty} h^1_i(G')))$.

But we know that $c^1_\rho(h^{*1}(G), h^{*1}(G')) = c^1_\rho(\lim_{i \to \infty} h^1_i(G), \lim_{i \to \infty} h^1_i(G')))$.
Therefore, $c^1_\rho(h^{*1}(G), h^{*1}(G')) \leq h^{*1}(F)$.

As mentioned above, by a similar argument we can show that $c^2_\rho(h^{*2}(G), h^{*2}(G')) \geq h^{*2}(F)$. From these two inequalities it follows that $h^*(F) \subseteq c_\rho(h^*(G), h^*(G'))$.

Now we prove that h^* satisfies the Decomposition postulate. We have to consider two cases. Let $H = F \wedge_\rho G$. The proof for $H = F \vee_\rho G$ will be similar. By definition of md_ρ, $md_\rho(h(H)) = [h^1(H), 1]$ for all formula functions h. As $md^1_\rho(h(H))$ and $md^2_\rho(h(H))$ we will denote the upper and the lower bounds of the $md_\rho(h(H))$ interval.

Now we consider the sequences $h^1_1(F), h^1_2(F), \ldots$ and $md^1_\rho(h_1(H)), md^1_\rho(h_2(H)), \ldots$.

As all h_is are formula functions, they satisfy the Decomposition postulate, i.e., $(\forall i)(h_i(F) \subseteq md_\rho(h_i(H))$. Therefore, the sequence $h^1_1(F), h^1_2(F), \ldots$ dominates the sequence $md^1_\rho(h_1(H)), md^1_\rho(h_2(H)), \ldots$. Therefore, $\lim_{i \to \infty} md^1_\rho(h_i(H)) \leq \lim_{i \to \infty} h^1_i(F)$.

However, as stated earlier, $(\forall i)(md^1_\rho(h_i(H)) = h^1_i(H))$. Therefore $\lim_{i \to \infty} md^1_\rho(h_i(H)) = \lim_{i \to \infty} h^1_i(H)$. From the latter equality we establish that $\lim_{i \to \infty} h^1_i(H) \leq \lim_{i \to \infty} h^1_i(F)$. But we know that $\lim_{i \to \infty} h^1_i(H) = h^{*1}(H)$ and similarly $\lim_{i \to \infty} h^1_i(F) = h^{*1}(F)$, which therefore yields $h^{*1}(H) \leq h^{*1}(F)$. Finally, we notice that $md^1_\rho(h^*(H)) = h^{*1}(H)$, i.e., $md^1_\rho(h^*(H)) \leq h^{*1}(F)$. As we know that $md^2_\rho(h^*(H)) = 1$ and $h^*(F) \leq 1$, we get the desired: $h^*(F) = [h^{*1}(F), h^{*2}(F)] \subseteq [h^{*1}(H), 1] = [md^1_\rho(h^*(H)), md^2_\rho(h^*(H))] = md_\rho(h^*(H))$, proving that h^* satisfies Decomposition and therefore proving the statement of the theorem. ■

As is common in lattice theory, we use $\sqcup H$ to denote the least upperbound of H.

Given any hp-program P, we wish to associate with P, an operator T_P that maps hybrid formula functions to hybrid formula functions. We do this by first defining a (similar) intermediate operator S_P that is used subsequently to define T_P.

Definition 18 *Let P be* a hybrid probabilistic program. *Operator $S_P : \mathcal{HFF} \longrightarrow \mathcal{HFF}$ is defined as follows (where F is a basic formula):*

$$S_P(h)(F) = \cap M \textit{ where}$$

$M = \{\mu\sigma | F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$ is a ground instance of some clause in P; *σ is a ground substitution of annotation variables and* $(\forall j \leq n) h(F_j) \subseteq \mu_j \sigma\}$ *if* $M \neq \emptyset$ *and* $S_P(h)(F) = [0, 1]$ *if if* $M = \emptyset$.

The operator S_P is very simple. Given $h \in \mathcal{HFF}$ and a basic formula F, it proceeds as follows: (i) First, it finds all ground instances of rules in P such that the head of the rule instance is of the form $F : \mu$ and

[2] I.e. $(\forall i)(h^1_i(F) \geq c^1_\rho(h^1_i(G), h^1_i(G')))$.

such that for each $F_j : \mu_j$ in the body, $h(F_j) \subseteq \mu_j$, i.e. h says that F_j's probability does in fact lie within the interval μ_j. (ii) It then takes the intersection of the intervals associated with the heads of all rules identified in the preceding step. Note that in the above definition, it is entirely possible that $S_P(h)(F)$ could be the empty set. In this case, there is an intuitive inconsistency, because the formula function $S_P(h)$ is saying that F's probability lies in the empty set. However, this is absurd, as the empty set cannot contain anything. This will be discussed in further detail in Section 4.2.

Example 7 *Consider our stock example.* Let h assign the following values to the atoms:

h(ch-sells-stock(c)) $= [0.8, 0.8]$

h(ch-retires(c)) $= [0.1, 0.1]$

h(strike(c)) $= [0.4, 0.5]$

h(price-drop(c)) $= [0.7, 0.9]$

h(stable(c)) $= [0.5, 0.6]$

Assume that for all other ground atoms A, $h(A) = [0, 1]$.

Now, suppose we want to compute $S_P(h)$(price-drop(c)). There are two ground rule instances with price-drop(c) as their head in the set of all groundizations of rules in P:

price-drop(c):$[0.4, 0.9] \longleftarrow$ (ch-sells-stock(c) $\vee_{igd}$ ch-retires(c)):$[0.6, 1]$.

price-drop(c):$[0.5, 1] \longleftarrow$ (strike(c) $\vee_{ind}$ accident(c)):$[0.3, 1]$.

First we compute

- h((strike(c) $\vee_{ind}$ accident(c)) $= c_{ind}(h$(strike(c)),accident(c)) $= c_{ind}([0.4, 0.5], [0, 1]) = [\min(1, 0.4 + 0 - 0.4 * 0), \min(1, 0.5 + 1 - 0.5 * 1)] = [0.4, 1] \subseteq [0.3, 1]$.
- h((ch-sells-stock(c) $\vee_{igd}$ ch-retires(c))) $= c_{igd}$(ch-sells-stock(c),ch-retires(c))$= c_{igd}([0.8, 0.8], [0.1, 0.1]) = [max(0.8, 0.1), min(1, 0.8 + 0.9)] = [0.8, 0.9] \subseteq [0.6, 1]$.

Since both rules will fire, $M = \{[0.4, 0.9], [0.5, 1]\}$ and therefore, $S_P(h)$(price-drop(c)) $= [0.4, 0.9] \cap [0.5, 1] = [0.5, 0.9]$.

However, the S_P operator is not quite "right." The reason is that in order to determine F's probability, it is not enough to merely look for rule instances whose head is identical to F. For instance, F might be $(p \wedge_{ig} q)$. The probability of $(p \wedge_{ig} q)$ may certainly be influenced by rules with head $p : \mu'$ because such rules may impose additional restrictions on p's probability – and hence on $(p \wedge_{ig} q)$'s probability. Thus, S_P, by itself, does not allow us to accurately infer the probability associated with a formula F. S_P needs to be *augmented appropriately* in order to do so. However, before augmenting S_P to create a new operator T_P that avoids this problem, we present a simple monotonicity property of S_P. *Note that S_P is monotonic regardless of what p-strategies occur in P.*

Lemma 1 S_P is monotonic, i.e., if h_1, h_2 are two formula functions and $h_1 \leq h_2$ then, $S_P(h_1) \leq S_P(h_2)$.

Proof.

Let F be a hybrid basic formula. We have $h_1(F) \leq h_2(F)$. By def. of S_P,

$$S_P(h_1)(F) = \cap M_1$$

$$M_1 = \{\mu | F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$$

$$\text{is a ground instance of some clause in } P \text{ ; } (\forall j \leq n) h_1(F_j) \subseteq \mu_j\}$$

Since $h_1(F_j) \subseteq \mu_j$ can be rewritten as $\mu_j \leq h_1(F_j)$, using transitivity of $\leq$, we obtain that for any ground instance $F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$ of a rule of program P, such that $\mu \in M_1$, $\mu \in M_2$, where

$$M_2 = \{\mu | F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$$

$$\text{is a ground instance of some clause in } P\ ; (\forall j \leq n) h_2(F_j) \subseteq \mu_j\}$$

and therefore, $M_1 \subseteq M_2$. Therefore, $S_P(h_2)(F) = \cap M_1 = (\cap M_1) \cap (M_2 - M_1) = S_P(h_1)(F) \cap (M_2 - M_1) \subseteq S_P(h_1)(F)$ i.e., $S_P(h_1)(F) \leq S_P(h_2)(F)$. □

Let us now define the T_P operator. Intuitively, the T_P operator builds on top of the S_P operator because the probability interval assignments made by the S_P operator to some formulas may allow us to derive sharper bounds for *other* formulas. However, these sharper bounds may not be found by the S_P operator. The T_P operator defined below takes such derivations into account.

Definition 19 *Let P be* a hybrid probabilistic program. We inductively define operator $T_P : \mathcal{HFF} \longrightarrow \mathcal{HFF}$ as follows:

1. *Let F be an* <u>*atomic*</u> *formula.*
   - *if $S_P(h)(F) = \emptyset$ then $T_P(h)(F) = \emptyset$.*
   - *if $S_P(h)(F) \neq \emptyset$, then let*

   $$M = \{\langle \mu\sigma, \rho\rangle | (F *_\rho G) : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$$

   where $* \in \{\vee, \wedge\}$ *and σ is a ground substitution of the annotation variables and $i \in \mathcal{S}$ and $(\forall j \leq n) h(F_j) \subseteq \mu_j\sigma\}$.* We define

   $$T_P(h)(F) = (\cap\{md_\rho(\mu\sigma) | \langle \mu\sigma, \rho\rangle \in M\}) \cap S_P(h)(F)$$

2. *(F* <u>*not atomic*</u>*) Let $F = F_1 *_\rho \ldots *_\rho F_n$.*
   Let $M' = \{\langle \mu\sigma, i\rangle | D_1 *_\rho \ldots *_\rho D_k : \mu \longleftarrow E_1 : \mu_1 \wedge \ldots E_m : \mu_m \in ground(P)$; $(\forall 1 \leq j \leq m) h(E_j) \subseteq \mu_j$; $\{F_1, \ldots, F_n\} \subset \{D_1, \ldots D_k\}, n < k\}$
   Then:

$$T_P(h)(F) = S_P(h)(F) \cap (\cap\{c_\rho(T_P(h)(G), T_P(h)(H)) | G \oplus H = F\}) \cap (\cap\{md_\rho(\mu\sigma) | \langle \mu\sigma, i\rangle \in M'\})$$

The intuition underlying the T_P operator is as follows: (i) Consider an atomic formula F: if $S_P(h)(F) = \emptyset$, then this means that an inconsistency (to be made more formal in Section 4.2) has occurred. For instance, if we have an hp-program containing two facts $a : [0,0]$ and $a : [1,1]$, then whatever h we pick, $S_P(h)(a) = \emptyset$, reflecting the (in this case flagrant) inconsistency in P. Thus, $T_P(h)$ must also assign $\emptyset$ to F. If $S_P(h)(F) \neq \emptyset$, then it may be case that $S_P(h)$ has assigned too "wide" an interval to F, because it ignores rules that are "associated" with F. As F is atomic, there might be rules whose bodies are satisfied by h, which include F in its head. We must find all such rules, and "split" the rule head into its F part, and the non-F part, say G. Clearly, the rule head must be of the form $(F *_\rho G)$ where $*$ is either $\wedge$ or $\vee$. As the rule's body is satisfied by h, it means that the head of this rule, viz. $(F *_\rho G)$ has probability in the interval μ. The rule in question thus allows us to conclude that F's probability ranges anywhere in $md_\rho(\mu)$ which is the "maximal interval" associated with F w.r.t. the connective $*_\rho$. We repeat this for each rule with F as part of the head.

(ii) When F is not a ground atom, there can be three sources of bounds on F's probability interval. The first source taken care of by the S_P operator are the rules with F as their head. The second source consists of information that can be inductively obtained by computing T_P for every pair G, H of formulas such that $G \oplus H = F$ (notice that we require both G and H to be non-empty), and using c_ρ to combine these values. Finally, some heads of the rules of the program may contain F as the proper subset. The probability range of F from each such rule is determined by the md_ρ function. Combining (intersecting) the ranges obtained from all three sources we obtain the final value of T_P operator.

Note that while the T_P operator is defined to be the intersection of many possible intervals, there are **at most two** intervals which will actually affect the final value of $T_P(h)$ for any particular formula F (one interval to provide the lower bound and one interval to provide the upper bound of $T_P(h)(F)$). Because of this, one can see that while the number of intervals to be intersected to obtain $T_P(h)(F)$ according to the definition above can be large, there is a simple *nondeterministic* algorithm that would perform this computation. This algorithm would guess how the two relevant intervals are obtained, and will only perform computations to produce these **two** intervals. This suggests that the problem of computing T_P is NP-complete — however, a detailed study of complexity issues in hybrid probabilistic programs is beyond the scope of this paper.

The following example demonstrates how T_P is computed.

Example 8 *Let us consider the stock program P and the formula function h from the previous example.* Suppose we want to compute $T_P(h)$(price-drop(c)$\wedge_{pcc}$buy-stock(c)) (i.e., the probability of the fact that the drop in price of stocks will result in purchases of new stock of company c).

It is easy to see that $T_P(h)$(price-drop(c)$\wedge_{pcc}$buy-stock(c))= $c_{pcc}(T_P(h)(\text{price-drop(c)}), T_P(h)(\text{buy-stock(c)}))$, as the heads of all rules in P are atomic.

$T_P(h)$(price-drop(c)) $=$ $S_P(h)$(price-drop(c)) $=$ $[0.5, 0.9]$ (see **Example 3**). $T_P(h)$(buy-stock(c)) $=$ $S_P(h)$(buy-stock(c)). To find the latter we consider the following ground rule in P :

buy-stock(c):$[0.7, 1] \longleftarrow$ (price-drop(c) $\wedge_{inc}$ stable(c)):$[0.3, 1]$.

Recall from **Example 3** that h(price-drop(c)) $= [0.7, 0.9]$ and h(stable(c)) $= [0.5, 0.6]$. Then, h((price-drop(c) $\wedge_{inc}$ stable(c))) $=$ $c_{inc}(h(\text{price-drop(c)}), h(\text{stable(c)}))$ $=$ $c_{inc}([0.7, 0.9], [0.5, 0.6])$ $=$ $[0.7 * 0.5, 0.9 * 0.6] = [0.35, 0.54] \subseteq [0.3, 1]$, which entails that $S_P(h)$(buy-stock(c))) $= [0.7, 1]$

Finally, $T_P(h)$(price-drop(c)$\wedge_{pcc}$buy-stock(c)= $c_{pcc}(T_P(h)(\text{price-drop(c)}), T_P(h)(\text{buy-stock(c)})$ $=$ $c_{pcc}([0.5, 0.9], [0.7, 1]) = [\min(0.5, 0.7), \min(0.9, 1)] = [0.5, 0.7]$. □

Let us consider another example:

Example 9 In this example we will consider a simple knowledge-base about the possible sales of three items: a, b and c. The unary predicate $s(X)$ is to be interpreted as "item X has been sold". Suppose the program P looks as follows:

$$\begin{array}{|l|}\hline s(a) \vee_{ind} s(b) \vee_{ind} s(c) : [0.4, 0.6] \longleftarrow . \\ s(a) \wedge_{igc} s(b) : [0, 0.5] \longleftarrow . \\ s(a) \wedge_{inc} s(c) : [\min(\frac{V}{2} + 0.1, \frac{W}{2}), \frac{W}{2}] \longleftarrow s(c) : [V, W] \\ s(c) : [0, 0.3] \longleftarrow . \\ \hline \end{array}$$

The first rule of the program states that the probability that at least one of the three items had been sold *under the assumption of independence* between possible sales is between 40% and 50% . The second rule states that the probability that *both* items a and b have been sold computed *under assumption of of ignorance* about the relationship of possible sales will be not more than 50% . The fourth rule just states that the probability that item c had been sold is no more than 30% .

Finally, the third rule of the program, states that if we know that the probability that item c had been sold is in the range $[V, W]$, then the probability that *both* items a and c have been sold, considered under the assumption of *independence* between possible sales, will be not more than $\frac{W}{2}$ and no less than the *minimum* of $\frac{V}{2} + 0.1$ and $\frac{W}{2}$.

Now let us look at how we can compute the T_P operator for this program. Let us take $h(F) = [0, 1]$ for all basic formulas F (i.e. our h is the bottom function $\perp$).

In this example we will be tracing all atomic formulas ($s(a)$, $s(b)$ and $s(c)$) as well as a few more complex formulas, such as $s(a) \vee_{ind} s(c)$, $s(a) \wedge_{igc} s(b)$ and $s(a) \wedge_{inc} s(c)$.

- First we have to compute $S_P(h)$. Clearly, we have the following:
  $S_P(h)(s(a)) = S_P(h)(s(b)) = [0,1]$
  $S_P(h)(s(c)) = [0,0.3]$
  $S_P(h)(s(a) \vee_{ind} s(c)) = [0,1]$
  $S_P(h)(s(a) \wedge_{igc} s(b)) = [0,0.5]$
  $S_P(h)(s(a) \wedge_{inc} s(c)) = [0.1,0.5]$

  Every $S_P(h)$ computation except for the last one is straightforward, since for each formula there is either only one rule (with an empty body) in the program that has it as its head, or there are no such rules at all. In the first case, the probability interval from the head of the rule gets to be the value of $S_P(h)$, in the second case, it will be $[0,1]$.
  The last computation requires more effort. Indeed, the third rule of our program is not ground (because of the variable annotation), therefore it will produce more than one ground instance. However, there will be only one ground instance of this rule which will have the body, "satisfied" by h:
  $$s(a) \wedge_{inc} s(c) : [0.1, 0.5] \longleftarrow s(c) : [0,1] \tag{1}$$
  since $h(s(c)) = [0,1]$ (we also point out that $\min(\frac{0}{2}+0.1, \frac{1}{2}) = \frac{0}{2}+0.1 = 0.1$). Therefore, $[0.1, 0.5]$ will be the value of $S_P(h)(s(c))$.
- Now let us compute the values of the T_P operator.
  1. $\underline{T_P(h)(s(a)).}$
     $s(a)$ appears in the heads of 3 rules of interest: first and second rules of the program and in the ground instance of the third rule shown above (1). This means that
     $$T_P(h)(s(a)) = md_{ind}([0.4,0.6]) \cap md_{igc}([0,0.5]) \cap md_{inc}([0.1,0.5]) =$$
     $$= [0,0.6] \cap [0,1] \cap [0.1,1] = [0.1,0.6]$$
  2. $\underline{T_P(h)(s(b)).}$
     $s(b)$ appears in the heads of first two rules of the program. Therefore:
     $$T_P(h)(s(b)) = md_{ind}([0.4,0.6]) \cap md_{igc}([0,0.5]) = [0,0.6] \cap [0,1] = [0,0.6]$$
  3. $\underline{T_P(h)(s(c)).}$ $s(c)$, besides constituting the head of the fourth rule of the program, is also a part of the heads of the first rule the program and rule (1). Applying the definition of the T_P operator here we obtain:
     $$T_P(h)(s(c)) = S_P(h)(s(c)) \cap md_{ind}([0.4,0.6]) \cap md_{inc}([0.1,0.5]) =$$
     $$= [0,0.3] \cap [0,0.6] \cap [0.1,0.5] = [0.1,0.3]$$
  4. $\underline{T_P(h)(s(a) \vee_{ind} s(c)).}$ $s(a) \vee_{ind} s(c)$ appears as a part of the head of the first rule of the program. By definition of T_P operator:
     $$T_P(h)(s(a) \vee_{ind} s(c)) = md_{ind}([0.4,0.6]) \cap c_{ind}(T_P(h)(s(a)), T_P(h)(s(c))) =$$
     $$= [0,0.6] \cap c_{ind}([0.1,0.6],[0.1,0.5]) = [0,0.6] \cap [0.1+0.1-0.1 \cdot 0.1, 0.6+0.5-0.6 \cdot 0.5] =$$
     $$= [0,0.6] \cap [0.19,0.8] = [0.19,0.6]$$

5. $\underline{T_P(h)(s(a) \wedge_{igc} s(b))}$.

$$T_P(h)(s(a) \wedge_{igc} s(b)) = S_P(h)(s(a) \wedge_{igc} s(b)) \cap c_{igc}(T_P(h)(s(a)), T_P(h)(s(b))) =$$

$$= [0, 0.5] \cap c_{igc}([0.1, 0.6], [0, 0.6]) = [0.0.5] \cap [\max(0, 0.1 + 0, -1), \min(0.6, 0.6)] =$$

$$= [0, 0.5] \cap [0, 0.6] = [0, 0.5]$$

6. $\underline{T_P(h)(s(a) \wedge_{inc} s(c))}$.

$$T_P(h)(s(a) \wedge_{inc} s(c)) = S_P(h)(s(a) \wedge_{inc} s(c)) \cap c_{inc}(T_P(h)(s(a)), T_P(h)(s(c))) =$$

$$= [0.1, 0.5] \cap c_{inc}([0.1, 0.6], [0.1, 0.3]) = [0.1, 0.5] \cap [0.1 \cdot 0.1, 0.6 \cdot 0.3] =$$

$$= [0.1, 0.5] \cap [0.01, 0.18] = [0.1, 0.18]$$

□

It follows immediately from the definition of the T_P operator that, for any program P, formula function h and formula F, $T_P(h)(F) \subseteq S_P(h)(F)$. The following result says that regardless of which p-strategies are considered in P, the T_P operator is guaranteed to be monotonic.

Theorem 4 T_P is monotonic,i.e., if h_1, h_2 are two formula functions and $h_1 \leq h_2$ then, $T_P(h_1) \leq T_P(h_2)$.

Proof.

Let F be a hybrid basic formula. We proceed by induction on $rank(F)$, i..e. number of atoms in the formula.

- F is an atomic formula. We have $h_1(F) \leq h_2(F)$. Let us assume that both $S_P(h_1)(F)$ and $S_P(h_2)(F)$ are non-empty. (Otherwise, we must have $S_P(h_2)(F) = \emptyset$ which implies $T_P(h_2)(F) = \emptyset$ and therefore, it must be the case that $T_P(h_2)(F) \subseteq T_P(h_1)(F)$). By lemma $S_P(h_1)(F) \leq S_P(h_2)(F)$. Let us consider

$$M_1 = \{\mu | (F *_\rho G) : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$$

$$\text{where } * \in \{\vee, \wedge\} \text{ and } \rho \in \mathcal{S} \text{ and } (\forall j \leq n) h_1(F_j) \subseteq \mu_j\}$$

Since $h_1(F_j) \subseteq \mu_j$ can be rewritten as $\mu_j \leq h_1(F_j)$, using transitivity of $\leq$, we obtain that for any ground instance $F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$ of a rule of program P, such that $\mu \in M_1$, $\mu \in M_2$, where

$$M_2 = \{\mu | (F *_\rho G) : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$$

$$\text{is a ground instance of some clause in } P \,; (\forall j \leq n) h_2(F_j) \subseteq \mu_j\}$$

Therefore, $M_1 \subseteq M_2$. But this means that $M_1' = \{md(\mu) | \mu \in M_1\} \subseteq M_2' = \{md(\mu) | \mu \in M_2\}$. Then $\cap M_2' \subseteq \cap M_1'$, i.e., $\cap M_1' \leq \cap M_2'$.

Since $T_P(h_1)(F) = S_P(h_1)(F) \cap (\cap M_1')$, $T_P(h_2)(F) = S_P(h_2)(F) \cap (\cap M_2')$, $S_P(h_1)(F) \leq S_P(h_2)(F)$ and $\cap M_1' \leq \cap M_2'$, we obtain that

$$T_P(h_1)(F) \leq T_P(h_2)(F)$$

- Let the theorem hold for all basic hybrid formulas of rank less than k. Let $rank(F) = k$ and $F = F_1 \vee_\rho \ldots \vee_\rho F_n$ or $F = F_1 \wedge_\rho \ldots \wedge_\rho F_n$.

From <u>Lemma 1</u> we know that $S_P(h_1)(F) \leq S_P(h_2)(F)$.

Let G, H be such formulas, that $G \oplus H = F$. By the induction hypothesis, (since $rank(G) < k$ and $rank(H) < k$, we have $T_P(h_1)(G) \leq T_P(h_2)(G)$ and $T_P(h_1)(H) \leq T_P(h_2)(H)$, therefore, by monotonicity axiom for p-strategies (applied twice) we have:

$$c_\rho(T_P(h_2)(G), T_P(h_2)(H)) \subseteq c_\rho(T_P(h_1)(G), T_P(h_1)(H))$$

i.e.

$$c_\rho(T_P(h_1)(G), T_P(h_1)(H)) \leq c_\rho(T_P(h_2)(G), T_P(h_2)(H))$$

From this is follows that

$$(\cap\{c_\rho(T_P(h_1)(G), T_P(h_1)(H)) | G \oplus H = F\}) \leq (\cap\{c_\rho(T_P(h_2)(G), T_P(h_2)(H)) | G \oplus H = F\})$$

Finally, let $M_1 = \{D_1 *_\rho \ldots *_\rho D_k : \mu \longleftarrow E_1 : \mu_1 \wedge \ldots E_m : \mu_m \in ground(P) | (\forall 1 \leq j \leq m) h_1(E_j) \subseteq \mu_j; \{F_1, \ldots, F_n\} \subset \{D_1, \ldots D_k\}, n < k\}$ and $M_2 = \{D_1 *_\rho \ldots *_\rho D_k : \mu \longleftarrow E_1 : \mu_1 \wedge \ldots E_m : \mu_m \in ground(P) | (\forall 1 \leq j \leq m) h_2(E_j) \subseteq \mu_j; \{F_1, \ldots, F_n\} \subset \{D_1, \ldots D_k\}, n < k\}$ Let $M_1' = \mu | D : \mu \longleftarrow Body \in M_1$ and $M_2' = \mu | D : \mu \longleftarrow Body \in M_2$

since $h_1 \leq h_2$, we can claim that if some ground instance $C \in M_1$, C also is in M_2, i.e., $M_1 \subseteq M_2$. Therefore $(\cap\{\mu | \mu \in M_2'\}) \subseteq (\cap\{\mu | \mu \in M_1'\})$, i.e., $(\cap\{\mu | \mu \in M_1'\}) \leq (\cap\{\mu | \mu \in M_2'\})$. Combining the established results into one, using the formula for $T_P(h)(F)$ we obtain the desired $T_P(h_1)(F) \leq T_P(h_2)(F)$. □

Again, note that the above result applies regardless of what set of p-strategies occur in program P. It is easy to see now that we may define the *iterations* of T_P as:

Definition 20
1. *$T_P^0 = h_\perp$ where $\perp$ is the atomic function that assigns $[0,1]$ to all ground atoms A.*
2. *$T_P^\alpha = T_P(T_P^{\alpha-1})$ where α is a successor ordinal whose predecessor is denoted by $\alpha - 1$.*
3. *$T_P^\gamma = \sqcup\{T_P^\alpha | \alpha < \gamma\}$, where γ is limit ordinal.*

In [8] it was established that if all clauses in P have only constant annotations then $lfp(T_P) = T_P^\omega$, where $lfp(T_P)$ is the least fixed point of T_P. This, however, turns out to not be the case when P has clauses with variable annotations. The following example is from [24].

Example 10 *([24])* Consider the program
$A : [0, V/2] \longleftarrow A : [0, V]$
$B : [0, 0] \longleftarrow A : [0, 0]$
The second rule of the program states that if the probability of A is known to be $[0,0]$ then the probability of B is also $[0,0]$. The first rule of the program states that if we know that the probability of A lies between 0 and some V, we may conclude that the probability of A lies in the bottom half ($[0, V/2]$) of the interval $[0, V]$.

Since $T_P^0(A) = [0,1]$, after the first iteration $T_P^1(A) = [0, 0.5]$. At each subsequent iteration, we will narrow the interval assigned to A by half. A T_P^ω assigns A the intersection of all T_P^l, $l < \omega$, it will assign $[0,0]$ interval to A. Then $T_P^{\omega+1}$ will finally assign $[0,0]$ to B.

Definition 21 *A hybrid probabilistic program P is said to satisfy the fixpoint reachability condition iff*

$$(\forall F \in bf_S(L))(\exists n < \omega)(lfp(T_P)(F) = T_P^n(F)).$$

Intuitively, if an hp-program P satisfies the fixpoint reachability condition, then this means that for every formula F, if $\mu \supseteq lfp(T_P)(F)$ is, then this means that there is a finitely long justification of this fact.

4.2 Probabilistic Model Theory

We are now ready to define a logical model theory for hp-programs. For this purpose, hybrid basic formula functions will play the role of an "interpretation." The key inductive definition of satisfaction is given below.

Definition 22 Satisfaction. Let h be hybrid basic formula function, $F \in bf_{\mathcal{S}}(B_L)$, $\mu \in \mathcal{C}[0,1]$. We say that

- $h \models F : \mu$ **iff** $h(F) \subseteq \mu$.
- $h \models F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$ **iff** $(\forall 1 \leq j \leq n) h \models F_j : \mu_j$.
- $h \models F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$ **iff** either $h \models F : \mu$ or $h \not\models F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$.
- $h \models (\exists x)(F : \mu)$ **iff** $h \models F(t/x) : \mu$ *for some ground term* t.
- $h \models (\forall x)(F : \mu)$ **iff** $h \models F(t/x) : \mu$ *for every ground term* t.

A formula function h is called a model *of an hp-program P ($h \models P$)* **iff** $(\forall p \in P)(h \models p)$.

As usual, we say that $F : \mu$ is a *consequence* of P iff for every model h of P, it is the case that $h(F) \subseteq \mu$.

Recall, from Section 4.1, that we can have cases where a hybrid formula function, h, could assign $\emptyset$ to some formula. When $h(F) = \emptyset$, h is "saying" that F's probability lies in the empty set. This corresponds to an inconsistency because, by definition, nothing is in the empty set.

Definition 23 *Formula function h is called* fully defined *iff*
$\forall (F \in bf_{\mathcal{S}}(B_L))(h(F) \neq \emptyset)$.

The following important result fully ties together the fixpoint theory associated with hp-programs, and the model theoretical characterization of hp-programs, regardless of which p-strategies occur in the hp-program being considered.

Theorem 5 *Let P be any hp-program. Then:*

1. *h is a model of P iff $T_P(h) \leq h$.*
2. *P has a model iff $lfp(T_P)$ is fully defined.*
3. *If $lfp(T_P)$ is fully defined, then it is the least model of P, and $F : \mu$ is a logical consequence of P iff $lfp(T_P)(F) \subseteq \mu$.*

Proof.

1. Claim 1. $T_P(h) \leq h \Longrightarrow h \models P$.
   Let $F \in bf_{\mathcal{S}}(B_L)$.
   Let $P' = \{p \in ground(P) | p$ is of form $F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n\}$.
   Two cases are possible. If $P' = \emptyset$ then P has no rules with F in the head and therefore $h \models P'$ by def.
   Let $P' \neq \emptyset$.
   Consider a rule $p' \in P'$. p' is of a form $F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$. Two cases are possible.
   - $(\forall\ 1 \leq j \leq n)(\mu_j \leq h(F_j))$ In this case, we know that
     $h \models F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$. We have to show that
     $h \models F : \mu$, i.e. $h(F) \subseteq \mu$.
     By our assumption, $T_P(h)(F) \leq h(F)$, i.e., $h(F) \subseteq T_P(h)(F)$. By definition of T_P and S_P operators, it is always the case that $T_P(h)(F) \subseteq S_P(h)(F)$. We now show that $S_P(h)(F) \subseteq \mu$.
     By definition, $S_P(h)(F) = \cap \mathcal{F}$ where $\mathcal{F} = \{\mu | F : \mu \longleftarrow F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$ is a ground instance of a rule in $P; (\forall\ 1 \leq j \leq n)(\mu_j \leq h(F_j))\}$. We know that $p' : \mu \in \mathcal{F}$, therefore, $S_P(h)(F) \subseteq \mu$, which implies that $T_P(h)(F) \subseteq \mu$. Combining together we obtain: $h(F) \subseteq T_P(h)(F) \subseteq S_P(h)(F) \subseteq \mu$ which implies $h \models F : \mu$, therefore, $h \models p'$.

- $(\exists\ 1 \leq j \leq n)(h(F_j) \not\subseteq \mu_j)$ in this case $h \not\models F_j : \mu_j$, therefore, $h \not\models F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$, and therefore, $h \models p'$.

This proves the first claim.

Claim 2. $h \models P \Longrightarrow T_P(h) \leq h$.

Let $F \in bf_{\mathcal{S}}(B_L)$. We prove the claim by induction on $rank(F)$.

- Base Case. $rank(F) = 0$, i.e., F is atomic. Let

$$F : \mu_1 \longleftarrow \ldots$$

$$\ldots$$

$$F : \mu_k \longleftarrow \ldots$$

$$(F *_{\rho_1} G_1) : \nu_1 \longleftarrow \ldots$$

$$\ldots$$

$$(F *_{\rho_m} G_1) : \nu_m \longleftarrow \ldots$$

  be the list of all rules from program P that contain F in the head, such that, h satisfies their bodies.

  By definition of T_P, $T_P(h)(F) = \mu_1 \cap \mu_2 \cap \ldots \cap \mu_k \cap md_{\rho_1}(\nu_1) \cap \ldots \cap md_{\rho_m}(\nu_m)$.

  Since h satisfies all the bodies of these rules, h must also satisfy all the heads, i.e., $(\forall\ 1 \leq \jmath \leq k)(h(F) \subseteq \mu_j)$ and $(\forall\ 1 \leq \jmath \leq l)(h(F *_{\rho_j} G) \subseteq \nu_j)$. From first set of inequalities we obtain: $h(F) \subseteq \mu_1 \cap \mu_2 \cap \ldots \cap \mu_k$.

  From second set of inequalities: $h(F *_{\rho_j} G) = c_{\rho_j}(h(F), h(G))$ and therefore $h(F) \subseteq md_{\rho_j}(\nu_j)$. This leads to $h(F) \subseteq md_{\rho_1}(\nu_1) \cap \ldots \cap md_{\rho_m}(\nu_m)$, which combined with previous result gives us desired $h(F) \subseteq T_P(h)(F)$ i.e., $T_P(h)(F) \leq h(F)$.

- Induction Step. Let our claim hold for all basic formulas rank less than k. Let $rank(F) = k$ and $F = A_1 *_\rho \ldots *_\rho A_k$.

  Let

$$F : \mu_1 \longleftarrow \ldots$$

$$\ldots$$

$$F : \mu_k \longleftarrow \ldots$$

  be all the rules with F as the head, such that h satisfies their bodies. We must therefore, conclude that for each of these rules h satisfies its head, i.e., $h(F) \subseteq \mu_1 \cap \mu_2 \cap \ldots \cap \mu_k = S_P(h)(F)$.

  Let now G and H be basic formulas such that $G \oplus H = F$. By definition, $rank(g) < k$ and $rank(H) < k$, therefore, by the induction hypothesis, $h(G) \subseteq T_P(h(G))$ and $h(H) \subseteq T_P(h(H))$. Since $G \oplus H = F$, $G *_\rho H \equiv F$ and therefore $h(F) = h(G *_\rho H) \subseteq c_\rho(h(G), h(H)) \subseteq c_\rho(T_P(h)(G), T_P(h)(F)$ (the last inequality is due to monotonicity property of composition function). Therefore we conclude that

$$h(F) \subseteq (\cap\{c_\rho(T_P(h)(G), T_P(h)(H)) | G \oplus H = F\})$$

  Now, let

$$(F *_\rho D_1) : \nu_1 \longleftarrow \ldots$$

$$\ldots$$

$$(F *_\rho D_s) : \nu_s \longleftarrow \ldots$$

be all the ground instances of rules in P such that h satisfies thier bodies and F is a part of their heads. Since $h \models P$, $h \models (F *_\rho D_1) : \nu_1, \ldots, h \models (F *_\rho D_s) : \nu_s$, i.e., $(\forall 1 \leq j \leq s)(h(F *_\rho D_j) \subseteq \nu_j)$. But we know that $h(F *_\rho D_j) \subseteq c_\rho(h(F), h(D_j)) \subseteq \nu_j$. For this to be true it must be the case that $h(F) \subseteq md_\rho(\nu_j)$. Therefore, $h(F) \subseteq md_\rho(\nu_1) \cap \ldots \cap md_\rho(\nu_s)$. Combining the three inequalities together we obtain:

$$h(h) \subseteq S_P(F) \cap (\cap \{c_\rho(T_P(h)(G), T_P(h)(H)) | G \oplus H = F\}) \cap (md_\rho(\nu_1) \cap \ldots \cap md_\rho(\nu_s)) = T_P(h)(F)$$

which proves the theorem.

2. Let $lfp(T_P)$ be fully defined. Since we know that $T_P(lfp(T_P)) = lfp(T_P)$, it is also the case that $T_P(lfp(T_P)) \leq lfp(T_P)$. According to *part 1* of this theorem, $lfp(T_P)$ is a model of P.
Assume now that P has a model h. By definition of a model, h is fully defined. We know that $T_P(h) \leq h$. By construction of $lfp(T_P)$, and because of the monotonicity of T_P operator $lfp(T_P) \leq T_P(h)$. Therefore $lfp(T_P) \leq h$. This means that for all basic formulas F, $h(F) \subseteq lfp(T_P)(F)$. Since h is fully-defined, $lfp(T_P)$ **has** to be fully defined too.
3. *Part 3* of this theorem is a direct corollary of *Part 2* and **theorem 2**. □

The second result above links consistency of P programs with the fully defined-ness property of $lfp(T_P)$. Specifically, if there is an integer i such that either S_P^i or T_P^i are not fully defined, then T_P^ω cannot be fully defined either, and hence, P would not have a model.

5 Proof Procedure

At this stage, we have provided a complete description of the logical consequences of an hp-program P. In this section, we develop three query processing procedures.

- The first query processing procedure (Section 5.2), termed hp-resolution, builds upon previous approaches of Ng and Subrahmanian [24] by first requiring that programs P be compiled to a new set, $CL(P)$. Queries are then processed by a process akin to linear input resolution, with the difference that clauses from $CL(P)$ may be considered input clauses. This process suffers from the major flaw that construction of $CL(P)$, which is based on computation of $lfp(T_P)$ is prohibitively expensive. Hence, we introduce another two more refutation procedures.
- The second procedure (Section 5.3), termed HR_P-refutations, is more pragmatic. Rather than requiring a compilation step when a query Q is posed, HR_P refutations allow relevant parts of $CL(P)$ to be dynamically constructed. This has two advantages over hp-refutations. First, hp-refutations often "lose" right at the beginning, as the compilation process may take a tremendous amount of time and space. This does not happen with HR_P-refutations. Second, HR_P-refutations only need a small part of $CL(P)$, not all of it, and this small part may be constructed as needed.
- The third procedure (Section 5.4), expands upon HR_P, to use tabling, as initially introduced in logic programming by Tamaki and Sato[34]. This procedure assumes that caches (or tables) are bounded a priori in size - a situation certainly true in practical implementations where tables cannot grow in an unbounded fashion. Furthermore, table management in probabilistic logic programs is much more complicated than in ordinary logic programming for many reasons. First, a query does not merely have a set of answers. Rather, a query has associated answer substitutions, each of which has an associated probability range. As computation proceeds, these ranges may get refined or sharpened – something that does not happen in classical logic program tables. Second, caches in our framework may contain basic formulas with associated probabilities. Such caches implicitly contain probability ranges for basic formulas implied by the cached formulas, as well as

basic formulas that imply the cached formulas. A third difference between our work and classical logic program tabling is that there are often many ways to update a table in the case of probabilistic logic programs. We define cache update strategies, and show several different such strategies. We show how HR_P-refutations may be extended with arbitrary cache update strategies.

Unlike classical resolution, when dealing with annotated conjunctions and disjunctions, unifiers may not be unique, as noted by Ng and Subrahmanian[24]. Before proceeding to describe our different notions of resolution, we summarize observation of [24] below as it is necessary for the further development of our proof procedures.

5.1 Unification in HPPs

As rules of clauses in hp-programs may contain annotated basic formulas, any notion of unification must be able to handle unification of annotated basic formulas. In this section, we recapitulate from [24, pps 175–179] how this may be done. The contents of this subsection are not new contributions.

Definition 24
- Θ is a *unifier* of annotated conjunctions $C_1 \equiv A_1 \wedge_\rho \ldots \wedge_\rho A_{n_1}$ *and* $C_2 \equiv B_1 \wedge_{\rho'} \ldots \wedge_{\rho'} A_{n_2}$ **iff** $\rho, \rho' \in \mathcal{CONJ}$ and $\rho = \rho'$ and $\{A_k\Theta | 1 \leq k \leq n_1\} = \{B_k\Theta | 1 \leq k \leq n_2\}$
- Θ is a *unifier* of annotated disjunctions $D_1 \equiv A_1 \vee_\rho \ldots \vee_\rho B_{n_1}$ and $D_2 \equiv B_1 \vee_{\rho'} \ldots \vee_{\rho'} B_{n_2}$ **iff** $\rho, \rho' \in \mathcal{DISJ}$ and $\rho = \rho'$ and $\{A_k\Theta | 1 \leq k \leq n_1\} = \{B_k\Theta | 1 \leq k \leq n_2\}$

In order to proceed we need to define a notion of maximally general unifier.

Definition 25 Let $U(C_1, C_2)$ denote the set of all unifiers of C_1 and C_2. Let $\Theta_1, \Theta_2 \in U(C_1, C_2)$.

1. $\Theta_1 \leq \Theta_2$ **iff** there exists a substitution γ, such that $\Theta_1 = \Theta_2\gamma$.
2. $\Theta_1 \equiv \Theta_2$ **iff** $\Theta_1 \leq \Theta_2$ and $\Theta_2 \leq \Theta_1$.
3. Let $[\Theta] = \{\Theta' \in U(C_1, C_2) | \Theta \equiv \Theta'\}$.
4. $[\Theta_1] \leq [\Theta_2]$ **iff** there exists such γ that $[\Theta_1] = [\Theta_2\gamma]$.
5. $[\Theta_1] < [\Theta_2]$ **iff** $[\Theta_1] \leq [\Theta_2]$ and $[\Theta_2] \neq [\Theta_1]$.

From the above definition, it is easy to see that $\equiv$ is an equivalence relation on elements of $U(C_1, C_2)$ and $\leq$ is a partial order on $\{[\Theta] | \Theta \in U(C_1, C_2)\}$. We can define a notion of maximally general unifier.

Definition 26 $\Theta \in U(C_1, C_2)$ is a maximally general unifier (max-gu) of C_1 and C_2 **iff** there is **no** such other unifier $\Theta' \in U(C_1, C_2)$ that $[\Theta] \leq [\Theta']$.

The proof of the following result is quite complex and is given in [24, Lemma 12,pps 176-179].

Lemma 2 *([24, Lemma 12,pps 176-179]) If two basic formulas are unifiable then they have a max-gu (not necessarily unique).*

5.2 hp-Resolution

In general, in the presence of basic formulas, just "straight" resolution is not sufficient for query processing. The reason is that to establish a basic formula, e.g. $(p \wedge_\rho q) : \mu$, we might need to *separately* prove $p : \mu_1$ and $q : \mu_2$ and then combine μ_1, μ_2 using the composition function associated with p-strategy ρ. There are two ways to do this: (i) allow resolution not against hp-clauses in P, but against hp-clauses in an *expanded* version of P, or (ii) introduce, in addition to resolution, new rules of inference corresponding the the "expansion" steps alluded above. Both cases are essentially equivalent from the point of view of completeness. In this section we discuss the former procedure, while the latter one will be described in detail in the next section.

First, we add to P all "tautologies". Any formula of the form $F : [0,1]$ is a tautology as F's probability certainly lies in the $[0,1]$ interval.

Definition 27 Let P be an hp-program. Then $REDUN(P)$ is defined as

$$REDUN(P) = P \cup \{A : [0,1] \longleftarrow \; | A \in B_L\}.$$

In addition to the above tautologies, we need to "merge" rules together and/or infer "implied" rules. For example, if one rule has $F_1 : \mu_1$ in the head, and another has $F_2 : \mu_2$ in the head, and these are unifiable via max-gu Θ, then these two rules may jointly provide some information on the probability of $(F_1 \wedge_\rho F_2)$ where ρ is some p-strategy. Likewise, if $(F_1 *_\rho F_2) : \mu'$ is in the head of some rule, then this rule certainly provides some information about F_1's probability, and F_2's probability. The closure of P, defined below, expands the rules in P by performing such merges and/or inferences.

Definition 28 Let P be an hp-program. Then $\mathrm{CL}(P)$ (closure of P) is defined as follows

- $\mathrm{CL}^0(P) = REDUN(P)$.
- 1. For each pair of clauses $F_1 : \mu_1 \longleftarrow Body_1$ and $F_2 : \mu_2 \longleftarrow Body_2 \in \mathrm{CL}^j(P)$, such that their heads F_1 and F_2 are unifiable via max-gu Θ add clause $(F_1 : \mu_1 \cap \mu_2 \longleftarrow Body_1 \wedge Body_2)\Theta$ to $\mathrm{CL}^{j+1}(P)$.
  2. For each clause $F_1 *_\rho F_2 : \mu \longleftarrow Body \in \mathrm{CL}^j(P)$ add the following two clauses to $\mathrm{CL}^{j+1}(P)$:
     - $F_1 : md_\rho(\mu) \longleftarrow Body$
     - $F_2 : md_\rho(\mu) \longleftarrow Body$
  3. For each two clauses $(A_1 *_\rho \ldots *_\rho A_k) : \mu_1 \longleftarrow Body_1$ and $(B_1 *_\rho \ldots *_\rho B_l) : \mu_1 \longleftarrow Body_2 \in \mathrm{CL}^j(P)$, $k > 1, l \geq 1$, add the clause

     $$(A_1 *_\rho \ldots *_\rho A_k *_\rho B_1 *_\rho \ldots *_\rho B_l) : c_\rho(\mu_1, \mu_2) \longleftarrow Body_1 \wedge Body_2$$

     to $\mathrm{CL}^{j+1}(P)$.
  4. if A and B are atoms, and $\mathrm{CL}^j(P)$ contains clauses $A : \mu_1 \longleftarrow Body_1$ and $B : \mu_2 \longleftarrow Body_2$, add

     $$(A *_\rho B) : c_\rho(\mu_1, \mu_2) \longleftarrow Body_1 \wedge Body_2$$

     for each $i \in \mathcal{CONJ} \cup \mathcal{DISJ}$ to $\mathrm{CL}^{j+1}(P)$.
- $\mathrm{CL}(P) = \bigcup_{j \geq 0} \mathrm{CL}^j(P)$

The following result says that the above steps are all sound. No new rule is produced that was not already a logical consequence of P.

Lemma 3 *For every clause $C \in CL(P)$, $P \models C$.*

Proof. Let C be a clause in $\mathrm{CL}(P)$. Then $C \in \mathrm{CL}^j(P)$ for some integer $j \geq 0$. We proceed by induction on j.

- Base Case.
  1. $C \in P$. Then by definition of $\models$, $P \models C$.
  2. $C \notin P$, $C \in REDUN(P)$. In this case C is of the form $A : [0,1] \longleftarrow$, and A is a ground instance of an atom. Let h be a formula function, such that $h \models P$. It is always the case that $h(C) \subseteq [0,1]$, which yields $h \models C$.
- Induction Step.

  Assume that for each clause $C \in \mathrm{CL}^j(P)$, $P \models C$. Let $C \in \mathrm{CL}^{j+1}(P) - \mathrm{CL}^j(P)$.

  As $C \in \mathrm{CL}^{j+1}(P) - \mathrm{CL}^j(P)$, C must have been inserted into $\mathrm{CL}^{j+1}(P)$ by the means of one of the cases $1-4$ from **Def. 26**. We have to consider each case separately.

1. Suppose C was inserted by the means of case 1. Then there exist such clauses $C_1 \equiv F_1 : \mu_1 \longleftarrow Body_1$ and $C_2 \equiv F_2 : \mu_2 \longleftarrow Body_2$, such that, $C_1 \in \mathrm{CL}^j(P), C_2 \in \mathrm{CL}^j(P)$, F_1 and F_2 are unifiable via max-gu Θ, and

$$C \equiv (F_1 : \mu_1 \cap \mu_2 \longleftarrow Body_1 \wedge Body_2)\Theta$$

We need to show that $P \models C$. Suppose h is a model of P, i.e., $h \models P$, and $C\gamma$ is a ground instance of C, such that $h \models (Body_1 \wedge Body_2)\Theta\gamma$. By the induction hypothesis, $h \models C_1$ and $h \models C_2$, therefore, $h \models C_1\Theta\gamma$ and $h \models C_2\Theta\gamma$. As $h \models Body_1\Theta\gamma$, we conclude that $h(F_1\Theta\gamma) \subseteq \mu_1$. Likewise we can conclude that $h(F_2\Theta\gamma) \subseteq \mu_2$.

But since Θ is a max-gu of F_1 and F_2, $F_1\Theta\gamma = F_2\Theta\gamma$, and therefore $h(F_1\Theta\gamma) \subseteq \mu_1 \cap \mu_2$, i.e. $h \models F_1\Theta\gamma : \mu_1 \cap \mu_2$.

2. Suppose C was inserted by the means of case 2. Then there exists such a clause $C_1 \equiv (F_1 *_\rho F_2) : \mu \longleftarrow Body \in \mathrm{CL}^j(P)$, that either

$$C \equiv F_1 : md_\rho(\mu) \longleftarrow Body$$

or

$$C \equiv F_2 : md_\rho(\mu) \longleftarrow Body.$$

We will consider the former case, the latter case is symmetric. We need to show that $P \models C$. Let $C\gamma$ be a ground instance of C and let $h \models P$ and $h \models Body\gamma$. By induction hypothesis, $h \models C_1$, and therefore, $h((F_1 *_\rho F_2)\gamma) \subseteq \mu$. By the definitions of md_ρ and h, this yields $h(F_1) \subseteq md_\rho(\mu)$, i.e., $h \models F_1 : md_\rho(\mu)$.

3. Let C be inserted by the means of case 3. In this case, $\mathrm{CL}^j(P)$ will contain two clauses, $C_1 \equiv (A_1 *_\rho \ldots *_\rho A_k) : \mu_1 \longleftarrow Body_1$ and $C_2 \equiv (B_1 *_\rho \ldots *_\rho B_l) : \mu_2 \longleftarrow Body_2$, such that, $k > 1, l \geq 1$, and

$$C \equiv (A_1 *_\rho \ldots *_\rho A_k *_\rho B_1 *_\rho \ldots *_\rho B_l) : c_\rho(\mu_1, \mu_2) \longleftarrow Body_1 \wedge Body_2.$$

We need to show $P \models C$. Let $C\gamma$ be a ground instance of C and let $h \models P$ and $h \models (Body_1 \wedge Body_2)\gamma$. By induction hypothesis, $h \models C_1$ and $h \models C_2$, and therefore, $h \models C_1\gamma$ and $h \models C_2\gamma$. Since $h \models Body_1\gamma$ and $h \models Body_2\gamma$, we have $h \models (A_1 *_\rho \ldots *_\rho A_k)\gamma : \mu_1$ and $h \models (B_1 *_\rho \ldots *_\rho B_l)\gamma : \mu_2$, i.e., $h((A_1 *_\rho \ldots A_k)\gamma) \subseteq \mu_1$, and $h((B_1 *_\rho \ldots *_\rho B_l)\gamma) \subseteq \mu_2$. But then,

$$h((A_1 *_\rho \ldots *_\rho A_k *_\rho B_1 *_\rho \ldots *_\rho B_l)\gamma) = c_\rho(h((A_1 *_\rho \ldots *_\rho A_k)\gamma), h((B_1 *_\rho \ldots *_\rho B_l)\gamma)) \subseteq c_\rho(\mu_1, \mu_2),$$

which means $h \models C$.

4. Finally, let C be inserted in $\mathrm{CL}^{j+1}(P)$ by the means of case 4. Then, $\mathrm{CL}^j(P)$ will contain 2 clauses, $C_1 \equiv A : \mu_1 \longleftarrow Body_1$ and $C_2 \equiv B : \mu_2 \longleftarrow Body_2$, such that both A and B are atomic, and

$$C \equiv (A *_\rho B) : c_\rho(\mu_1, \mu_2) \longleftarrow Body_1 \wedge Body_2$$

for some p-strategy ρ.

We have to show $P \models C$. Let $C\gamma$ be a ground instance of C and let $h \models P$ and $h \models (Body_1 \wedge Body_2)\gamma$. By induction hypothesis, $h \models C_1$ and $h \models C_2$, therefore, $h \models C_1\gamma$ and $h \models C_2\gamma$. Since $h \models Body_1\gamma$ and $h \models Body_2\gamma$, we obtain $h \models A\gamma : \mu_1$ and $h \models B\gamma : \mu_2$, i.e. $h(A\gamma) \subseteq \mu_1$ and $h(B\gamma) \subseteq \mu_2$. Hence, $h((A *_\rho B)\gamma) = c_\rho(h(A\gamma), h(B\gamma)) \subseteq c_\rho(\mu_1, \mu_2)$, which means that $h \models C\gamma$ and therefore $h \models C$. □

We now present a refutation procedure for query processing.

Definition 29 A query is a formula of the form $\exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$, where $(\forall 1 \leq i \leq n)$ $(F_i \in bf_{\mathcal{S}}(B_L))$. F_is need not be ground.

Definition 30 Suppose $C \equiv G : \lambda \longleftarrow G_1 : \lambda_1 \wedge \ldots \wedge G_m : \lambda_m \in \mathrm{CL}(P)$ and $Q \equiv \exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$ is a query. Let C and Q be standardized apart. Let also G and F_i be unifiable for some $1 \leq i \leq n$. Then

$$\exists((F_1 : \mu_1 \wedge \ldots \wedge F_{i-1} : \mu_{i-1} \wedge G_1 : \lambda_1 \wedge \ldots \wedge G_m : \lambda_m \wedge F_{i+1} : \mu_{i+1} \wedge \ldots \wedge F_n : \mu_n)\Theta)$$

is an hp-resolvent of C and Q **iff**:

1. Θ is a max-gu of G and F_i
2. $\lambda\Theta$ and $\mu_i\Theta$ are ground and $\lambda\Theta \subseteq \mu_i\Theta$

If Θ is a unifier but not necessarily a max-gu, we call the resolvent an unrestricted hp-resolvent.

Definition 31 Let $Q \equiv \exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$ be an initial query, and P an hp-program. An hp-deduction of Q from P is a sequence $< Q_1, C_1, \Theta_1 > \ldots < Q_r, C_r, \Theta_r > \ldots$ where, $Q = Q_1$, for all $i \geq 1$, C_i is a renamed version of a clause in $\mathrm{CL}(P)$ and Q_{i+1} is an *hp-resolvent* of Q_i and C_i via max-gu Θ_i.
If the Θ_i's are not restricted to be max-gu's, we call the resulting sequence an unrestricted hp-deduction.

Definition 32 Let $Q \equiv \exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$ be an initial query, and P an hp-program. An hp-refutation of Q from P is a *finite hp-deduction* $< Q_1, C_1, \Theta_1 > \ldots < Q_r, C_r, \Theta_r >$ where, the hp-resolvent of Q_r and C_r via Θ_r is the empty query. $\Theta_1 \ldots \Theta_r$ is called the *computed answer substitution.*

We are now in a position to state the soundness and completeness of hp-resolution.

Theorem 6 (Soundness of hp-refutation).
Let P be an hp-program, and Q be an initial query. If there exists an hp-refutation of $Q \equiv \exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$ from P with the answer substitution Θ then $P \models \forall((F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)\Theta)$.

Proof. Let $< Q_1, C_1, \Theta_1 > \ldots < Q_n, C_n, \Theta_n >$ be our hp-refutation. We proceed by induction on n.

Base case: $n = 1$

In this case $Q_1 \equiv F_1 : \mu_1$, $C_1 \equiv G_1 : \nu_1 \longleftarrow \in \mathrm{CL}(P)$, $F_1\Theta_1 = G_1\Theta_1$ and $\nu_1 \subseteq \mu_1$. Let $h \models P$. By the previous lemma, $h \models C_1$. Therefore, $h \models \forall(G_1 : \nu_1)$ and in particular $h \models \forall((G_1 : \nu_1)\Theta_1)$. But, since $F_1\Theta_1 = G_1\Theta_1$ and $\nu_1 \subseteq \mu_1$, we get $h \models \forall(F_1 : \mu_1)\Theta_1)$.

Induction Step.

Suppose the theorem holds for any hp-refutation $< Q_2, C_2, \Theta_2 > \ldots < Q_n, C_n, \Theta_n >$. Consider an hp-refutation $< Q_1, C_1, \Theta_1 >, < Q_2, C_2, \Theta_2 > \ldots < Q_n, C_n, \Theta_n >$. Let $h \models P$. Let $Q_1 \equiv (F_1 : \mu_1 \wedge \ldots \wedge F_m : \mu_m)$ and $C_1 \equiv G : \nu \longleftarrow Body$ be (a renamed version of) a clause in $\mathrm{CL}(P)$, such that for some $1 \leq i \leq m$, $F_i\Theta_1 = G\Theta_1$ and $\nu \subseteq \mu_i$. Then, $Q_2 \equiv (F_1 : \mu_1 \wedge \ldots \wedge F_{i-1} : \mu_{i-1} \wedge Body \wedge F_{i+1} : \mu_{i+1} \wedge \ldots \wedge F_m : \mu_m)\Theta_1$. By induction hypothesis, $h \models \forall(Q_2\Theta_2 \ldots \Theta_n)$, i.e. $h \models \forall(F_1 : \mu_1 \wedge \ldots \wedge F_{i-1} : \mu_{i-1} \wedge Body \wedge F_{i+1} : \mu_{i+1} \wedge \ldots \wedge F_m : \mu_m)\Theta_1\Theta_2 \ldots \Theta_n)$. Therefore, $h \models \forall(F_1 : \mu_1 \wedge \ldots \wedge F_{i-1} : \mu_{i-1} \wedge F_{i+1} : \mu_{i+1} \wedge \ldots \wedge F_m : \mu_m)\Theta_1\Theta_2 \ldots \Theta_n)$ and $h \models \forall(Body\Theta_1 \ldots \Theta_n)$. Since also $h \models C_1$, we obtain $h \models \forall((G\Theta_1 \ldots \Theta_n) : \nu)$. Since $\nu \subseteq \mu_i$, we obtain $h \models \forall((F_i\Theta_1 \ldots \Theta_n) : \mu_i))$, i.e. $h \models \forall((F_i : \mu_i)\Theta_1 \ldots \Theta_n)$. Combining with $h \models \forall(F_1 : \mu_1 \wedge \ldots \wedge F_{i-1} : \mu_{i-1} \wedge F_{i+1} : \mu_{i+1} \wedge \ldots \wedge F_m : \mu_m)\Theta_1\Theta_2 \ldots \Theta_n)$ we get the desired: $h \models \forall(Q_1\Theta_1 \ldots \Theta_n)$, i.e., $h \models \forall(Q_1\Theta)$. □

In order to establish completeness, we need some intermediate facts. The following two lemmas can be proved by a straightforward application of mgu and lifting lemmas for classical logic programming in Lloyd[23].

Lemma 4 *(***Max-gu Lemma***). Let Q be a query that has an* unrestricted *hp-refutation from an hp-*

program P. Then, Q has an hp-refutation of the same length and if $\Theta_1, \ldots, \Theta_m$ are the unifiers form the unrestricted hp-refutation, and $\Theta'_1, \ldots, \Theta'_m$ are the max-gu's from the hp-refutation, then, for some γ $\Theta_1 \ldots \Theta_m = \Theta'_1, \ldots, \Theta'_m \gamma$.

Lemma 5 *(***Lifting Lemma***) Let P be an hp-program, Q be a query, Θ be a substitution. Let $Q\Theta$ have an hp-refutation from P. Then Q has an hp-refutation from P of the same length. Also, if $\Theta_1, \ldots, \Theta_m$ are the max-gu's from the refutation of $Q\Theta$ and $\Theta'_1, \ldots, \Theta'_m$ are the max-gu's from the refutation of Q then, for some substitution γ: $\Theta\Theta_1 \ldots \Theta_m = \Theta'_1, \ldots, \Theta'_m \gamma$.*

Using the above two results, we may now establish a completeness result for hp-refutations. The proof is long and arduous.

Theorem 7 (Completeness of hp-refutation).
Let P be a consistent hp-program which satisfies the fixpoint reachability condition (see Def. 21) and Q' be a query. Then, if $P \models \exists(Q')$ then there exists an hp-refutation of Q' from P.

Proof.

Since $P \models \exists(Q')$, there exists such a ground substitution Θ that $P \models Q'\Theta$. Let $Q \equiv Q'\Theta$. We will prove that Q has an hp-refutation from P. By Lifting Lemma, Q' will also have a refutation from P.

Let $Q \equiv F_1 : \mu_1 \wedge \ldots \wedge F_m : \mu_m$. Since $P \models Q$, it must be the case that $P \models F_i : \mu_i$, $1 \leq i \leq m$.

Claim 1: *Let $F : \mu$ and $G : \nu$ be ground annotated formulas which have hp-refutations from P. Then, so does $F : \mu \wedge G : \nu$.*

Proof. Let $< F : \mu, C_1, \Theta_1 > \ldots < Q_l^F, C_l, \Theta_l >$ be the hp-refutation for $F : \mu$. Let $< G : \nu, D_1, \Gamma_1 >$ $\ldots < Q_k^G, D_k\Gamma_k >$ be the hp-refutation for $G : \nu$. Then, as $F : \mu$ and $G : \nu$ are ground, the following will be the hp-refutation for $F : \mu \wedge G : \nu$:

$$< F : \mu \wedge G : \nu, C_1, \Theta_1 >, < Q_2^F \wedge G : \nu, C_2, \Theta_2 > \ldots < Q_l^F \wedge G : \nu, C_l, \Theta_l >, < G : \nu, D_1, \Gamma_1 > \ldots < Q_k^G, D_k\Gamma_k >$$

. This completes the proof of **Claim 1**.

Now all we have to prove is

Claim 2:*Let $P \models \exists(F' : \mu)$. Then there exists a refutation of $F' : \mu$ from P.*

Proof. Since $P \models F' : \mu$, there exists a ground substitution θ such that $P \models F'\theta : \mu$. Let $F = F'\theta$. We show that $F : \mu$ has an hp-refutation from P, and by *lifting lemma* so will $F' : \mu$.

Since $P \models F : \mu$, by **Theorem 3**, $T_P^\omega(F) \subseteq \mu$. By definition of T_P^ω, there exists such an $\alpha < \omega$ that $T_P^\alpha(F) \subseteq \mu$. Consider the smallest such integer. We now proceed by induction on α.

Base Case: $\alpha = 0$. By definition of T_P^0, $T_P^0(F) = [0, 1]$. Therefore, $\mu = [0, 1]$.

If F is atomic, then, since F is ground, a clause

$$C \equiv F : [0, 1] \longleftarrow$$

is in $REDUN(P)$, and therefore it is in $\mathrm{CL}(P)$. Then, $< F : \mu, C, e >$ (e is the empty substitution) is the hp-refutation for $F : \mu$.

Let $F \equiv (A_1 *_\rho A_2 *_\rho \ldots *_\rho A_n)$, where each A_i is atomic. Then, a set of clauses

$$C_i \equiv A_j : [0, 1] \longleftarrow$$

is in $REDUN(P)$ and therefore each of these clauses is in $\mathrm{CL}^0(P)$. By definition of $\mathrm{CL}(P)$ and because for any p-strategy ρ $c_\rho([0, 1], [0, 1]) = [0, 1]$, $\mathrm{CL}^n(P)$ (and therefore $\mathrm{CL}(P)$) will contain the clause

$$C \equiv (A_1 *_\rho A_2 *_\rho \ldots *_\rho A_n) : [0, 1] \longleftarrow$$

(In fact we can argue that the above clause will be contained in $\mathrm{CL}^{\log_2(n)}(P)$).

Then the refutation for $F : \mu$ will be $< F : \mu, C, e >$ (e is the empty substitution).

Induction Step. Assume that for any formula $G : \mu$ such that $T_P^{\alpha-1} \models G : \mu$, there exists a refutation ξ of $G : \mu$ from P.

We prove the claim by induction on the structure of F.

Base Case. F is atomic.

Let $M' = \{\mu' | G : \mu' \longleftarrow Body \in P; T_P^{\alpha-1} \models Body$, where G is unifiable with $F\}$. We notice that $S_P(T_P^{\alpha-1})(F) = \cap\{\mu' | \mu' \in M'\}$.

Let $M'' = \{\mu'' | (G *_\rho H) : \mu'' \longleftarrow Body \in P; T_P^{\alpha-1} \models Body$, where G is unifiable with $F\}$.

We have, by definition of T_P^α ($\alpha > 0$):

$$T_P^\alpha(F) = S_P(T_P^{\alpha-1})(F) \cap (\cap\{md(\mu'') | \mu'' \in M''\}) \subseteq \mu$$

.

Two cases are possible.

1. $|M' \cup M''| = 1$.
   Assume $M' \neq \emptyset$. Than, there is a unique rule $C' \equiv G : \mu' \longleftarrow Body \in P$, s.t., G unifies with F, $T_P^{\alpha-1} \models Body$, and $S_P(T_P^{\alpha-1})(F) = \mu'$. (Notice that $C' \in P$ implies $C' \in \mathrm{CL}(P)$). Let Θ' be the max-gu for G and F.
   By induction hypothesis, there exists an hp-refutation $< Body, C_1, \Theta_1 > \ldots < Q_k, C_k, \Theta_k >$ for *Body*. Then
   $$< F : \mu, C', \Theta' >, < Body, C_1, \Theta_1 > \ldots < Q_k, C_k, \Theta_k >$$
   is the refutation for $F : \mu$.
   Assume now that $M'' \neq \emptyset$. Then there is a unique rule $C' \equiv (G *_\rho H) : \mu' \leftarrow Body \in P$, s.t., G unifies with F via max-gu Θ', $T_P^{\alpha-1} \models Body$ and $T_P(T_P^{\alpha-1})(F) = md_\rho(\mu')$. Since $C' \in P$, $C' \in \mathrm{CL}^0(P)$ and therefore the following clause $C'' \equiv G : md_\rho(\mu') \longleftarrow Body$ is in $\mathrm{CL}^1(P)$. By the induction hypothesis, there exists an hp-refutation for *Body*: $< Body, C_1, \Theta_1 > \ldots < Q_k, C_k, \Theta_k >$. Then
   $$< F : \mu, C'', \Theta' >, < Body, C_1, \Theta_1 > \ldots < Q_k, C_k, \Theta_k >$$
   is the refutation for $F : \mu$.
2. $|M' \cup M''| > 1$
   Let $\mathcal{C}' = \{G : \mu' \longleftarrow Body \in P | T_P^{\alpha-1} \models Body\}$, where G is unifiable with F, and
   $M' = \{\mu' | G : \mu' \longleftarrow Body \in \mathcal{C}'\}$.
   Let also $\mathcal{C}'' = \{(D *_\rho H) : \mu'' \longleftarrow Body \in P | T_P^{\alpha-1} \models Body\}$, where D is unifiable with F and $M'' = \{\mu'' | (D *_\rho H) : \mu'' \longleftarrow Body \in \mathcal{C}''\}$.
   Since all clauses from $\mathcal{C}'$ are in P, they are also in $\mathrm{CL}^0(P)$. Let
   $$G_1 : \mu_1' \longleftarrow Body_1'$$
   $$\ldots$$
   $$G_s : \mu_s' \longleftarrow Body_s'$$
   be all clauses in $\mathcal{C}'$. Since they are in $\mathrm{CL}^0(P)$, we can claim that the clause
   $$C_1 \equiv G_1\Theta' : \mu_1' \cap \ldots \cap \mu_s' \longleftarrow Body_1' \wedge \ldots \wedge Body_s'$$

will be in $\mathrm{CL}^s(P)$ (actually, it will already be in $\mathrm{CL}^{\log_2(s)}(P)$) where Θ' is the max-gu of $G_1,\ldots,G_s$ (such a substitution must exist since we know that each of G_j is unifiable with F).
Let

$$(D_1 *_{\rho_1} H_1) : \mu''_1 \longleftarrow Body''_1$$

$$\ldots$$

$$(D_r *_{\rho_r} H_r) : \mu''_r \longleftarrow Body''_r$$

be all clauses in $\mathcal{C}''$. Since $\mathcal{C}'' \subseteq P$, every clause in $\mathcal{C}''$ is also in $\mathrm{CL}^0(P)$. Therefore, the following set of clauses:

$$D_1 : md_{\rho_1}(\mu''_1) \longleftarrow Body''_1$$

$$\ldots$$

$$D_r : md_{\rho_r}(\mu''_r) \longleftarrow Body''_r$$

will be a subset of $\mathrm{CL}^1(P)$. Then, we can claim that $\mathrm{CL}^r(P)$ (or even $\mathrm{CL}^{\log_2(r)}(P)$ will contain the following clause:

$$C_2 \equiv D_1\Theta'' : md_{\rho_1}(\mu''_1) \cap \ldots \cap md_{\rho_r}(\mu''_r) \longleftarrow Body''_1 \wedge \ldots \wedge Body''_r$$

where Θ'' is the max-gu for $D_1,\ldots D_r$.

Let $l = \max(r,s)$. Since both $C_1 \in \mathrm{CL}^l(P)$ and $C_2 \in \mathrm{CL}^l(P)$, the following clause

$$C \equiv G_1\Theta'_1 : \mu'_1 \cap \ldots \cap \mu'_s \cap md_{\rho_1}(\mu''_1) \cap \ldots \cap md_{\rho_r}(\mu''_r) \longleftarrow Body'_1 \wedge \ldots \wedge Body'_s \wedge Body''_1 \wedge \ldots \wedge Body''_r$$

(where Θ'_1 is the max-gu of $G_1\Theta'$ and $D_1\Theta''$) will be in $\mathrm{CL}^{l+1}(P)$ and therefore, in $\mathrm{CL}(P)$.
Notice that $\mu'_1 \cap \ldots \cap \mu'_s \cap md_{\rho_1}(\mu''_1) \cap \ldots \cap md_{\rho_r}(\mu''_r) = T_P^\alpha(F) \subseteq \mu$. Also, by induction hypothesis, each $Body'_j$ and $Body''_j$ has an hp-refutation, therefore by **Claim 1** of the theorem, their conjunction has an hp-refutation. Let $< Q_1, C_1, \Theta_1 >, \ldots, < Q_z, C_z, \Theta_z >$ be such an hp-refutation. Than the following is an hp-refutation for $F : \mu$:

$$< F : \mu, C, \Gamma >, < Q_1, C_1, \Theta_1 >, \ldots, < Q_z, C_z, \Theta_z >$$

where Γ is the max-gu of F and $G_1\Theta'_1$.

Induction Step. Assume that the theorem holds for every formula of size less than k and let $F = A_1 *_\rho \ldots *_\rho A_k$, where $A_1,\ldots A_k$ are atomic.

Let $\mathcal{C}_1 = \{G : \mu' \longleftarrow Body \in P | T_P^{\alpha-1} \models Body$, where G is unifiable with $F\}$, and
$M_1 = \{\mu' | G : \mu' \longleftarrow Body \in \mathcal{C}_2\}$.

Let $\mathcal{C}_2 = \{(D *_\rho E) : \mu'' \longleftarrow Body \in P | T_P^{\alpha-1} \models Body$, where D is unifiable with $F\}$ and $M_2 = \{\mu'' | (D *_\rho E) : \mu'' \longleftarrow Body \in \mathcal{C}_2\}$.

Let

$$G_1 : \mu'_1 \longleftarrow Body'_1$$

$$\ldots$$

$$G_s : \mu'_s \longleftarrow Body'_s$$

be all clauses in $\mathcal{C}_1$. Since they are in $\mathrm{CL}^0(P)$, we can claim that the clause

$$C_1^F \equiv G_1\Theta' : \mu'_1 \cap \ldots \cap \mu'_s \longleftarrow Body'_1 \wedge \ldots \wedge Body'_s$$

will be in $\mathrm{CL}^s(P)$ (actually, it will already be in $\mathrm{CL}^{\log_2(s)}(P)$) where Θ' is the max-gu of $G_1, \ldots, G_s$ (such a substitution must exist since we know that each of G_j is unifiable with F).

Let

$$(D_1 *_{\rho_1} E_1) : \mu''_1 \longleftarrow Body''_1$$

$$\ldots$$

$$(E_r *_{\rho_r} E_r) : \mu''_r \longleftarrow Body''_r$$

be all clauses in $\mathcal{C}_2$. Since $\mathcal{C}_2 \subseteq P$, every clause in $\mathcal{C}_2$ is also in $\mathrm{CL}^0(P)$. Therefore, the following set of clauses:

$$D_1 : md_{\rho_1}(\mu''_1) \longleftarrow Body''_1$$

$$\ldots$$

$$D_r : md_{\rho_r}(\mu''_r) \longleftarrow Body''_r$$

will be a subset of $\mathrm{CL}^1(P)$. Then, we can claim that $\mathrm{CL}^r(P)$ (or even $\mathrm{CL}^{\log_2(r)}(P)$ will contain the following clause:

$$C_2^F \equiv D_1\Theta'' : md_{\rho_1}(\mu''_1) \cap \ldots \cap md_{\rho_r}(\mu''_r) \longleftarrow Body''_1 \wedge \ldots \wedge Body''_r$$

where Θ'' is the max-gu for $D_1, \ldots D_r$.

Now, consider any pair of basic formulas H and I such that $H \oplus I = F$. Since $F \equiv (H *_\rho I)$, we must conclude that $T_P^\alpha(F) = T_P^\alpha(H *_\rho I) = c_\rho(T_P^\alpha(H), T_P^\alpha(I))$. By our assumption $T_P^\alpha(F) \subseteq \mu$ therefore, $c_\rho(T_P^\alpha(H), T_P^\alpha(I)) \subseteq \mu$. Let $\nu' = T_P^\alpha(H), \nu'' = T_P^\alpha(I)$. We can now say that $T_P^\alpha \models H : \nu_1$ and $T_P^\alpha \models I : \nu_2$, such that $c_\rho(\nu_1, \nu_2) \subseteq \mu$.

By the induction hypothesis, there exist hp-refutations for $H : \nu'$ and $I : \nu''$. Let

$$< H : \nu', C_1^H, \Theta_1^H >< Q_2^H, C_2^H, \Theta_2^H > \ldots < Q_t^H, C_t^H, \Theta_t^H >$$

and

$$< I : \nu'', C_1^I, \Theta_1^I >< Q_2^I, C_2^I, \Theta_2^I > \ldots < Q_u^I, C_u^I, \Theta_u^I >$$

be these respective hp-refutations. Let us look at the clauses C_1^H and C_1^I. These clauses have to be of a (respective) form:

$$C_1^H \equiv H' : \lambda' \longleftarrow Body'$$

where, $\lambda' \subseteq \nu'$, $T_P^{\alpha-1} \models Body'\prime$, $H\prime$ is unifiable with H, and

$$C_1^I \equiv I' : \lambda'' \longleftarrow Body''$$

where, $\lambda' \subseteq \nu'$, $T_P^{\alpha-1} \models Body''$, I' is unifiable with I.

By definition of hp-refutation, both C_1^H and C_1^I are in $\mathrm{CL}(P)$. Let w be the smallest integer such that both $C_1^I \in \mathrm{CL}^w(P)$ and $C_1^H \in \mathrm{CL}^w(P)$. Then we can claim that $\mathrm{CL}^{w+1}(P)$ will contain the following clause:

$$C^{H *_\rho I} \equiv (H' *_\rho I') : c_\rho(\lambda', \lambda'') \longleftarrow Body' \wedge Body''$$

Since both $Body'$ and $Body''$ have hp-refutations, so does $Body' \wedge Body''$. In fact, we know that $< Q_2^H, C_2^H, \Theta_2^H > \ldots < Q_t^H, C_t^H, \Theta_t^H >$ is an hp-refutation for $Body'$ ($Q_2^H = Body'$) and $< Q_2^I, C_2^I, \Theta_2^I >$ $\ldots < Q_u^I, C_u^I, \Theta_u^I >$ is an hp-refutation for $Body''$ ($Q_2^I = Body''$. Then, the following will be an hp-refutation for $(H' *_\rho I') : c_\rho(\lambda', \lambda'')$:

$$< (H' *_\rho I') : c_\rho(\lambda', \lambda''), C^{H *_\rho I}, \Theta >, < Body' \wedge Body'', C_2^H, \Theta_2^H > \ldots$$
$$< Q_t^H \wedge Body'', C_t^H, \Theta_t^H >, < Body'', C_2^I, \Theta_2^I > \ldots, < Q_u^I, C_u^I, \Theta_u^I >$$

Let now $\mathcal{HI}\{< H_1, I_1 >, \ldots < H_m, I_m >\}$ be *all* possible pairs of basic formulas such that for each $< H, I > \in \mathcal{HI}$ $H \oplus I = F$. By applying the reasoning above we will conclude that for each pair $< H_j, I_j >$ CL(P) contains a clause

$$C_j \equiv (H'_j *_\rho I'_j) : \lambda_j \longleftarrow Body_j$$

that $\lambda_j \subseteq \mu$, $(H'_j *_\rho I'_j)$ is unifiable with F and H_j is unifiable with H'_j and I_j is unifiable with I'_j, $T_P^{\alpha-1} \models Body_j$. Let $q = \max\{q_1, \ldots q_m\}$, where $(\forall 1 \leq j \leq m)(C_j \in \mathrm{CL}^{q_j}(P)$ and $C_j \notin \mathrm{CL}^{q_j - 1}(P))$. Then $\mathrm{CL}^{q+m}(P)$ will contain the clause

$$C_3^F \equiv F\Theta_3^F : \lambda_1 \cap \ldots \cap \lambda_m \longleftarrow (Body_1 \wedge \ldots Body_m)\Theta_3^F$$

where Θ_3^F is the max-gu of $(H_1 *_\rho I_1), \ldots, (H_m *_\rho I_m)$. Since all $Body_1, \ldots Body_m$ have hp-refutations, so does $Body_1 \wedge \ldots Body_m$ (and therefore $Body_1 \wedge \ldots Body_m)\Theta_3^F$).

Now we can combine clauses C_1^F, C_2^F and C_3^F together into:

$$C^F \equiv F\Theta^F : \mu' \cap \mu'' \cap \lambda \longleftarrow Body^1 \wedge Body^2 \wedge Body^3$$

where Θ^F is a max-gu of the heads of C_1^F, C_2^F and C_3^F, μ',μ'' and λ are probability ranges of C_1^F, C_2^F and C_3^F respectively and $Body^1$, $Body^2$ and $Body^3$ are their respective bodies. It is clear that (i) $\mu' \cap \mu'' \cap \lambda \subseteq \mu$ and (ii) $C^F \in \mathrm{CL}(P)$. We also know that there exists an hp-refutation $< Body^1 \wedge Body^2 \wedge Body^3, C_1^B, \Theta_1^B > \ldots < Q_v^B, C_v^B, \Theta_v^B >$ of $Body^1 \wedge Body^2 \wedge Body^3$. Then the following is an hp-refutation for $F : \mu$:

$$< F : \mu, C^F, \Theta >, < Body^1 \wedge Body^2 \wedge Body^3, C_1^B, \Theta_1^B > \ldots < Q_v^B, C_v^B, \Theta_v^B >,$$

where Θ is a max-gu unifier of F and the head of C^F.

This completes the proof of the completeness theorem. □

It is important to note that the above theorem only holds when P satisfies the fixpoint reachability condition. Past work on annotated logics [24] make this assumption, and as this paper generalizes [24, 18], it is not possible to remove this assumption.

The hp-refutation paradigm extends a proof procedure developed in [24] for probabilistic logic programs under the ignorance assumption. In particular, Items (4) and (5) in the definition of $CL(P)$ given in Definition 28 do not occur in the proof procedure in [24]. The need for these two rules derives directly from the use of arbitrary p-strategies. This causes the proof procedure of [24] to be much simpler (and easier to implement) than that given in this paper.

5.3 HR_P Refutations for HP-programs

Note that the hp-refutation procedure assumes that $CL(P)$ has been constructed prior to processing a query. In practice, this is an extremely expensive process, both in terms of time taken to construct $CL(P)$, and in terms of space requirements. Even for propositional programs it is easy to see that $CL(P)$ can contain exponentially many clauses. Thus, in practice, constructing $CL(P)$ before a refutation is attempted, is often completely infeasible in practice. To avoid this *a priori* computation of $CL(P)$, we provide a new procedure that allows that part of $CL(P)$ needed in a refutation to be dynamically computed on an "as-needed" basis. The HR_P refutation framework described here avoids the construction of $CL(P)$.

In the definition below, anytime a formula $(F *_\rho G)$ is written it is assumed that $* \in \{\wedge, \vee\}$ and if $* = \wedge$ then $\rho \in \mathcal{CONJ}$ and if $* = \vee$ then $\rho \in \mathcal{DISJ}$.

Definition 33 *Let P be an hp-program. We define the <u>formal system HR_P</u> as follows:*

1. *<u>Axioms</u> of HR_P include all clauses from P and all clauses of the form: $A : [0,1] \longleftarrow$ where $A \in B_L$.*
2. *<u>Inference Rules</u>. There are 5 inference rule schemes in HR_P.*
   - *<u>A-Composition</u>: Let $A_1, A_2 \in B_L$*

     $$\frac{A_1 : \mu_1 \longleftarrow Body_1 \qquad A_2 : \mu_2 \longleftarrow Body_2}{(A_1 *_\rho A_2) : c_\rho(\mu_1, \mu_2) \longleftarrow Body_1 \wedge Body_2}$$

   - *<u>F-Composition</u>: Let $A_1, \ldots A_k, B_1 \ldots B_k \in B_L$*

     $$\frac{(A_1 *_\rho \ldots *_\rho A_k) : \mu_1 \longleftarrow Body_1 \qquad (B_1 *_\rho \ldots *_\rho B_l) : \mu_2 \longleftarrow Body_2}{(A_1 *_\rho \ldots *_\rho A_k *_\rho B_1 *_\rho \ldots *_\rho B_l) : c_\rho(\mu_1, \mu_2) \longleftarrow Body_1 \wedge Body_2}$$

   - *<u>Decomposition</u>:* *<u>L-Decomposition</u>* $\frac{(F *_\rho G):\mu \longleftarrow Body}{F : md_\rho(\mu) \longleftarrow Body}$ *<u>R-Decomposition</u>* $\frac{(F *_\rho G):\mu \longleftarrow Body}{G : md_\rho(\mu) \longleftarrow Body}$
   - *<u>Clarification</u>:*

     $$\frac{F_1 : \mu_1 \longleftarrow Body_1 \qquad F_2 : \mu_2 \longleftarrow Body_2}{(F_1 : \mu_1 \cap \mu_2 \longleftarrow Body_1 \wedge Body_2)\Theta}$$

     if F_1 and F_2 are unifiable via max-gu Θ
   - *<u>Exchange</u>: Let $A_1, \ldots, A_k \in B_L$, and let $B_1, \ldots, B_k$ be a* **permutation** *of $A_1, \ldots, A_k$*

     $$\frac{(A_1 *_\rho \ldots *_\rho A_k) : \mu \longleftarrow Body}{(B_1 *_\rho \ldots *_\rho B_k) : \mu \longleftarrow Body}$$

3. *A finite sequence $C_1 \ldots C_r$ of hp-clauses is called an* $\underline{HR_P\textbf{-derivation}}$ iff *each clause C_j is either* an axiom *or can be constructed from one or more previous of $C_1 \ldots C_{j-1}$ by applying one of the* inference rule schemes *to them. We call clause C_r <u>the result</u> of the HR_P-derivation.*
4. *An hp-clause C is derivable in HR_P* **iff** *there exists such an HR_P-derivation $C_1, \ldots C_r$ that $C_r = C$. We denote it by $P \vdash C$.*

The following theorems tell us that the system of axioms and inference rules describing HR_P precisely captures the closure, $CL(P)$, of P.

Theorem 8 *(soundness of HR_P w.r.t $CL(P)(P)$) For each hp-clause C, if $P \vdash C$ then $C \in CL(P)$.*

Proof. We notice first that the set of all axioms of HR_P is exactly $P \cup REDUN(P) = \mathrm{CL}^0(P)$. Next we notice that the first 4 inference rule schemes precisely match the 4 rules used to add new hp-rules to $\mathrm{CL}(P)$. Finally, the last inference rule scheme (*Exchange*) does not create a new basic formula, it just rearranges the order of atoms in it. □

Theorem 9 *(completeness of HR_P w.r.t. $CL(P)$) For each hp-clause C, if $C \in CL(P)$ then $P \vdash C$.*

Proof. if $C \in \mathrm{CL}(P)$ then there exists such an integer n that $C\mathrm{CL}^n(P)$ and $C \notin \mathrm{CL}^{n-1}(P)$. We prove the theorem using induction on n.

In the base case, $n = 0$ and we know that $\mathrm{CL}^0(P) = P \cup REDUN(P)$. As stated in the preceding theorem, this set is exactly the set of all axioms of HR_P, therefore, C is an axiom of HR_P.

On the induction step, we consider a clause C added to $\mathrm{CL}^n(P)$. By **Def 26** C was added to $\mathrm{CL}^n(P)$ by one of four rules. Since these rules match exactly the four inference rules of HR_P and by induction hypothesis for every clause $C' \in \mathrm{CL}^{n-1}(P)$ we know that $P \vdash_{HR_P} C'$, we can obtain the proof of C in HR_P by application of a matching rule to the same clauses. $\square$

Definition 34 (HR_P-refutations) Let $Q \equiv \exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$ be an initial query, and P an hp-program. An hp-refutation via HR_P of Q from P is a *finite* sequence $< Q_1, C_1, \Theta_1 > \ldots < Q_r, C_r, \Theta_r >$ where,

- $Q_1 = Q$
- Q_r is empty
- $P \vdash C_i$ for all $1 \leq i \leq r$
- Q_{i+1} is an hp-resolvent of Q_i and C_i with max-gu Θ_ρ, for all $1 \leq i < r$.

The following results tell us that hp-refutations using HR_P are both sound and complete and thus,they constitute the first sound and complete proof procedure for probabilistic logic programs (including those in [24]) that do not require the construction of a program closure. Here is a simple example of HR_P-refutations.

Example 11 (HR_P refutations) Consider the HP-program P given by:

$$\boxed{\begin{array}{l} a : [1,1] \longleftarrow (b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1] \wedge f : [0.5, 0.9]. \\ e : [1,1] \longleftarrow (b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1] \wedge f : [0.5, 1]. \\ (f \wedge_{ind} g) : [0.7, 0.8] \longleftarrow b : [1,1]. \\ (f \vee_{ig} g) : [0.7, 0.9] \longleftarrow . \\ b : [1,1] \longleftarrow (c \wedge_{ind} d) : [0.3, 1]. \\ c : [0.6, 1] \longleftarrow . \ d : [0.5, 1] \longleftarrow . \end{array}}$$

A refutation of the query $Q = a[0.9, 1] \wedge e : [1,1]$ is given by:

$Q_1 = Q = a[0.9, 1] \wedge e : [1,1]$
$P \ni C_1 = a : [1,1] \longleftarrow (b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1] \wedge f : [0.5, 0.9]$.

$Q_2 = (b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1] \wedge f : [0.5, 0.9] \wedge e : [1,1]$
$P \vdash C_2 = (b \wedge_{ind} c \wedge_{ind} d) : [0.3, 1] \longleftarrow (c \wedge_{ind} d) : [0.3, 1]$.

$Q_3 = (c \wedge_{ind} d) : [0.3, 1] \wedge f : [0.5, 0.9] \wedge e : [1,1]$
$P \vdash C_3 = (c \wedge_{ind} d) : [0.3, 1]) \longleftarrow$

$Q_4 = f : [0.5, 0.9] \wedge e : [1,1]$
$P \vdash C_4 = f : [0.7, 0.9] \longleftarrow b : [1,1]$.

$Q_5 = b : [1,1] \wedge e : [1,1]$
$P \ni C_5 = b : [1,1] \longleftarrow (c \wedge_{ind} d) : [0.3, 1]$.

$Q_6 = (c \wedge_{ind} d) : [0.3, 1] \wedge e : [1,1]$
$P \vdash C_6 = (c \wedge_{ind} d) : [0.3, 1]) \longleftarrow$

$Q_7 = e : [1,1]$
$P \ni C_7 = e : [1,1] \longleftarrow (b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1] \wedge f : [0.5, 0.1]$.

$Q_8 = (b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1] \wedge f : [0.5, 0.1]$.
$P \vdash C_8 = (b \wedge_{ind} c \wedge_{ind} d) : [0.3, 1] \longleftarrow (c \wedge_{ind} d) : [0.3, 1]$.

$Q_9 = (c \wedge_{ind} d) : [0.3, 1] \wedge f : [0.5, 0.1]$.
$P \vdash C_9 = (c \wedge_{ind} d) : [0.3, 1]) \longleftarrow$

$Q_{10} = f : [0.5, 0.1]$.
$P \vdash C_{10} f : [0.7, 0.9] \longleftarrow b : [1,1]$.

$Q_{11} = b : [1, 1]$.
$P \ni C_{11} = b : [1, 1] \longleftarrow (c \wedge_{ind} d) : [0.3, 1]$.

$Q_{12} = (c \wedge_{ind} d) : [0.3, 1]$.
$P \vdash C_{12} = (c \wedge_{ind} d) : [0.3, 1]) \longleftarrow$

$Q_{13} = \Box$

Here is another example of an HR_P refutation, for a program that contains variable annotations.

Example 12 Let us consider the program from Example 9. For convenience we repeat P below:

$$\boxed{\begin{array}{l} s(a) \vee_{ind} s(b) \vee_{ind} s(c) : [0.4, 0.6] \longleftarrow . \\ s(a) \wedge_{igc} s(b) : [0, 0.5] \longleftarrow . \\ s(a) \wedge_{inc} s(c) : [\min(\frac{V}{2} + 0.1, \frac{W}{2}), \frac{W}{2}] \longleftarrow s(c) : [V, W] \\ s(c) : [0, 0.3] \longleftarrow . \end{array}}$$

Let us now look at how the refutation of the query $Q = (s(a) \wedge_{inc} s(c)) : [0.15, 0.15]$ will proceed:

$Q_1 = Q = (s(a) \wedge_{inc} s(c)) : [0.15, 0.15]$
$P \vdash C_1 = s(a) \wedge_{inc} s(c) : [0.15, 0.15] \longleftarrow s(c) : [0.15, 0.3]$

$Q_2 = s(c) : [0.15, 0.3]$
$P \vdash C_2 = s(c) : [0.15, 0.3] \longleftarrow s(c) : [0.1, 0.3]$

$Q_3 = s(c) : [0.1, 0.3]$
$P \vdash C_3 = s(c) : [0.1, 0.3] \longleftarrow s(c) : [0, 0.6]$

$Q_4 = s(c) : [0, 0.6]$
$P \ni C_4 = s(c) : [0, 0.3] \longleftarrow .$

$Q_5 = \Box$.

Here is how the derivations of C_1, C_2 and C_3 are done:

- Rule $C_1 = s(a) \wedge_{inc} s(c) : [0.15, 0.15] \longleftarrow s(c) : [0.15, 0.3]$ is a ground instance of the rule
$$s(a) \wedge_{inc} s(c) : [\min(\frac{V}{2} + 0.1, \frac{W}{2}), \frac{W}{2}] \longleftarrow s(c) : [V, W]$$
with $V = 0.15$ and $W = 0.3$ ($\min(\frac{0.15}{2} + 0.1, \frac{0.3}{2}) = \frac{0.3}{2} = 0.15$).
- $C_2 = s(c) : [0.15, 0.3] \longleftarrow s(c) : [0.1, 0.3]$ is derived as follows: $C_2' = s(a) \wedge_{inc} s(c) : [0.15, 0.15] \longleftarrow s(c) : [0.1, 0.3]$ is a ground instance of the rule
$$s(a) \wedge_{inc} s(c) : [\min(\frac{V}{2} + 0.1, \frac{W}{2}), \frac{W}{2}] \longleftarrow s(c) : [V, W]$$
with $V = 0.15$ and $W = 0.3$. Applying the inference rule of *R-Decomposition* to C_2' we obtain $C_2'' = s(c) : [0.15, 1] \longleftarrow s(c) : [0.1, 0.3]$. Combining C_2'' with the rule $C_2^* = s(c) : [0, 0.3] \longleftarrow . \in P$ using the inference rule of *Clarification* we obtain C_2 as $[0, 0.3] \cap [0.15, 1] = [0.15, 0.3]$.
- The derivation of $C_3 = s(c) : [0.1, 0.3] \longleftarrow s(c) : [0, 0.6]$ is similar to the derivation of C_2: $C_3' = s(a) \wedge_{inc} s(c) : [0.1, 0.3] \longleftarrow s(c) : [0, 0.6]$ is a ground instance of the rule
$$s(a) \wedge_{inc} s(c) : [\min(\frac{V}{2} + 0.1, \frac{W}{2}] \longleftarrow s(c) : [V, W]$$
where $V = 0$ and $W = 0.6$ ($\min(\frac{0}{2} + 0.1, \frac{0.6}{2}) = \frac{0}{2} + 0.1 = 0.1$). Applying the inference rule of *R-Decomposition* to C_3' we obtain the rule $C_3'' = s(c) : [0.1, 1] \longleftarrow s(c) : [0, 0.6]$. Combining C_3'' with $C_3^* = s(c) : [0, 0.3] \longleftarrow . \in P$ using the inference rule of *Clarification* we obtain C_3 as $[0, 0.3] \cap [0.1, 1] = [0.1, 0.3]$.

The soundness and completeness of HR_P-refutations follow immediately from the soundness and completeness theorems for HP-refutation and soundness and completeness theorems for HR_P w.r.t. CL(P).

Theorem 10 (Soundness of HR_P)-Refutations).
Let P be an hp-program, and Q be an initial query. If there exists an hp-refutation via HR_P of $Q \equiv \exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$ from P with the answer substitution Θ then $P \models \forall((F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)\Theta)$.

Proof. Suppose $\langle Q_1, C_1, \Theta_1\rangle, \ldots, \langle Q_r, C_r, \Theta_r\rangle$ is an HR_P refutation of Q. Let $\mathbf{C} = \{C_i \mid C_i \notin P\}$. By bullet (3) in the definition of HR_P-refutations, it follows that $P \vdash C_i$ for all $C_i \in \mathbf{C}$. By Theorems 8, we know each such C_i is in $CL(P)$, and hence, $\langle Q_1, C_1, \Theta_1\rangle, \ldots, \langle Q_r, C_r, \Theta_r\rangle$ is an hp-refutation. By the soundness of hp-refutation (Theorem 6), the result follows. □

Theorem 11 (Completeness of HR_P)-Refutations).
Let P be a consistent hp-program which satisfies the fixpoint reachability conditions and Q' be a query. Then, if $P \models \exists(Q')$ then there exists an hp-refutation of Q' from P via HR_P.

Proof. By the completeness of hp-refutations (Theorem 7), it follows that there exists an hp-refutation of Q' from P. Suppose $\langle Q_1, C_1, \Theta_1\rangle, \ldots, \langle Q_r, C_r, \Theta_r\rangle$ is such an hp-refutation. Then, by definition of hp-refutations, each C_i is in $CL(P)$. But then, by Theorem 9, each C_i is either in P, or is such that $P \vdash C$, and hence, $\langle Q_1, C_1, \Theta_1\rangle, \ldots, \langle Q_r, C_r, \Theta_r\rangle$ is also an HR_P refutation. □

Before concluding this section, we briefly reiterate that HR_P refutations avoid compile-time construction of $CL(P)$ – an expensive and time/space consuming process.

5.4 B-Cache

We are now ready to study efficient tabled query processing techniques for HPPs. In this section, we will first define *caches* and *bounded caches*. Intuitively, a cache contains formulas with established probability ranges. As resolution based processing of a query occurs, we will gain information about certain basic formulas. These will need to be "added" to the cache. For this purpose, we will define in this section, a family of *updating strategies* and introduce several example strategies. Later, in Section 5.5, we will show how to use these tables and table update strategies hand in hand with the resolution based proof procedure.

5.4.1 Definitions

Definition 35 *A <u>cache</u> is a finite set of annotated basic formulas. If b is an integer, a <u>bounded b-cache</u> is a finite set of annotated basic formulas containing at most b atoms each.*

Basically a b-cache is a collection of hybrid probabilistic basic formulas, where each formula's length is bounded by a constant b. Note that a b-cache may be considered to be a hybrid probabilistic logic program all of whose clauses are "facts".

Definition 36 *Let T be a b-cache, F be a basic formula (not necessarily ground). By $T[F]$ we denote the set of all such pairs $\{< \mu, \Theta >\}$, where Θ is a substitution for F and $\mu \subseteq [0,1]$ is the smallest interval such that $T \models \forall(F\Theta : \mu)$.*

Intuitively $T[F]$ represents what the b-cache T "thinks" about the possible probability ranges of instances of F. Note that if F is ground, then $\{\mu | < \mu, \Theta > \in T[F]\}$ is a singleton set. Without loss of generality we will abuse notation in this case and write $T[F] = \mu$.

5.4.2 B-Cache Update strategies

We fix an integer $b > 0$, a logical language L as defined in Lloyd[23], and a set $\mathcal{S}$ of p-strategies. Let $\mathcal{T}[b, L, \mathcal{S}]$ denote the set of all possible b-caches over $bf_S(B_L)$. Whenever b, L and $\mathcal{S}$ are clear from the context we may use $\mathcal{T}$ instead of $\mathcal{T}[b, L, \mathcal{S}]$.

We are interested in developing a resolution-based query processing procedure that is irredundant in the sense that it does not "re-infer" facts that it has already inferred. In the case of classical logic programs, caches and their utilization are relatively simple: caches contain facts; when performing resolution on an atom A in the query, we check to see if A is subsumed by the cache (Tamaki and Sato[34]). An alternative approach is due to Warren et. al. who check the cache for variants of A [9, 7]. However, in the case of probabilistic logic programs, b-caches are somewhat more complicated.

As the resolution triggered by a query proceeds, more and more information is being established and any time new information is obtained, we want to insert it into our b-cache. However simple addition of a new basic formula to T is not enough, because as we add new probabilistic information - we might be able to update the probability intervals for some other basic formulas already in T. Also, the way such an update can be defined is not unique - in fact, there is a variety of possible "intuitive" updates.

Rather than defining a specific update procedure, we first proceed by defining the notion of an *update strategy* - a function that takes a b-cache and a basic formula as input, and returns a new "improved" b-cache. We will establish a number of basic properties of *any* update strategy. Later we will define a number of specific update strategies that are "natural" or "intuitive".

In the definition below, if S is a set of hp-formulas, $CN(S)$ denotes the set of all logical consequences of S.

Definition 37 *A function $f : \mathcal{T} \times bf_{\mathcal{S}}(B_L) \times \mathcal{C}[0,1] \longrightarrow \mathcal{T}$ is called a b-cache update strategy* **iff** *it satisfies the following conditions:*

1.
$$(\forall T \in \mathcal{T})(\forall F \in bf_{\mathcal{S}}(B_L))(\forall \mu \in \mathcal{C}[0,1]) CN(T) \subseteq CN(f(T,F,\mu)) \subseteq CN(T \cup \{F : \mu\})$$

2.
$$(\forall T \in \mathcal{T})(\forall F, G \in bf_{\mathcal{S}}(B_L))(\forall \mu, \nu \in \mathcal{C}[0,1]) f(f(T,F,\mu),G,\nu) = f(f(T,G,\nu),F,\mu)$$

3.
$$(\forall T \in \mathcal{T})(\forall F \in bf_{\mathcal{S}}(B_L))(\forall \mu \in \mathcal{C}[0,1]) f(f(T,F,\mu),F,\mu) = f(T,F,\mu)$$

We use the symbol $\uplus$ to denote b-cache update functions. When more than one update function is considered, we will use the $\uplus_f$ notation and annotate F with μ. (so, $f(T,F,\mu) = T \uplus_f F : \mu$).

Clause (1) in the above definition says that an update of a b-cache (i) should *not* decrease the amount of information that is contained in, or that can be deduced from the b-cache, but at the same time (ii) may not increase the content of the table "unreasonably". Notice that b-caches, by their very definition, automatically pose certain restrictions on how complete the update is - if the length of an updating formula is greater than b - the formula itself cannot be stored in the b-cache.

Clause (2) of the above definition says that the order in which we apply the update operator f should not matter. Updating a table T with $F : \mu$ first and then $G : \nu$ should be the same as doing it the other way around.

Finally, Clause (3) states that "redundant" updates should not change the b-cache.

Definition 38 *Let P be an hp-program and T be a b-cache. We say that T is <u>sound</u> w.r.t. P ($P \models T$) iff for each formula $F : \mu \in T$, $P \models F : \mu$.*

Lemma 6 *(soundness of b-cache update). Let P be an hp-program, T be a b-cache and F be a basic formula. Let f be any b-cache update strategy. Then if $P \models T$ and $P \models F : \mu$ then also*
$P \models T \uplus_f \{F : \mu\}$.

Proof. Let $F' : \mu' \in T \uplus_f F : \mu$. Two cases are possible.

1. $F' : \mu' \in T$. In this case, since $P \models T$, it has to be $P \models F' : \mu'$.

2. $F' : \mu' \notin T$. We know that $T \uplus_f F : \mu \models F' : \mu'$, hence $F' : \mu' \in CN(T \uplus_f F : \mu)$. We also know that $CN(T \uplus_f F : \mu) \subseteq CN(T \cup \{F : \mu\})$, therefore, we can obtain that $T \cup \{F : \mu\} \models F' : \mu'$. But, $P \models T$ and $P \models F : \mu$ implies that $P \models T \cup \{F : \mu\}$. Combining the obtained results together we get $P \models F' : \mu'$. □

In order to simplify notation we define an update of a b-cache with a finite set of formulas as follows:

Definition 39 *Let $S = \{F_1 : \mu_1, \ldots, F_n : \mu_n\}$ be a finite set of annotated basic formulas and u- a b-cache update strategy. We define*

$$T \uplus_u S = (\ldots(T \uplus_u F_1 : \mu) \uplus_u \ldots) \uplus_u F_n : \mu_n)$$

The order in which we write F_is is irrelevant as by the second property of the b-cache update strategy (commutativity), the result of updating a b-cache with a sequence of basic formulas does not depend on the order of formulas. (Second property establishes it for a sequence of 2 basic formulas. It is easily extended onto the case of sequences of 3 or more formulas).

As the reader may notice, there are numerous functions that satisfy the definition of an update strategy. Some of these are intuitively "more complete" than others. The following definition captures this informal notion.

Definition 40 *Let u and w be two b-cache update strategies. We say that u is <u>more complete</u> than w (denoted $u \geq w$)* **iff** $(\forall T \in \mathcal{T})(\forall F \in bf_{\mathcal{S}}(B_L))(\forall \mu \in \mathcal{C}[0,1]) CN(T \uplus_w F : \mu) \subseteq CN(T \uplus_u F : \mu)$.
Two update strategies u and w are <u>equivalent</u> **iff** *if both $u \geq w$ and $w \geq u$.*
An update strategy u is <u>maximally complete</u> **iff** $(\forall w)(u \geq w)$.

As pointed out earlier, Clause (3) of the definition of a b-cache update strategy postulates that no change in b-cache should occur when an update is repeated. However, this is not the only possible *redundant update*. The following proposition tells us how b-cache update strategies handle other *redundant* updates:

Proposition 2 *For any b-cache T, and b-cache update strategy f, any basic formula F and any interval $\mu \in \mathcal{C}[0,1]$ the following holds: if $T \models F : \mu$ then $CN(T) = CN(T \uplus_f F : \mu)$.*

Proof. Since $T \models F : \mu$, $CN(T) = CN(T \cup \{F : \mu\})$. Since $CN(T \uplus_f F : \mu) \subseteq CN(T \cup \{F : \mu\})$ and $CN(T) \subseteq CN(T \uplus_f F : \mu)$ we obtain the desired equality.

□

5.4.3 Examples of Update Strategies

In this section, we provide examples of a number of different update strategies, and show how these strategies are related to one another w.r.t. the "more complete" relationship.

The first kind of update strategy we consider is a relatively simple "atomic update."

Definition 41 (Atomic Updates) *Let T be a b-cache and A be an atomic (not necessarily ground) formula. An <u>atomic update</u> of T by $A : \mu$, denoted $T \uplus_{at} \{A : \mu\}$ is defined as follows:*

1. *If T has no atomic formulas that <u>unify</u> with $A : \mu$, then $T \uplus_{at} \{A : \mu\} = T \cup \{A : \mu \longleftarrow\}$*
2. *Otherwise we proceed in a number of steps:*
   - *(a) If there is a formula $A : \nu$ in T, we replace it with $A : \mu \cap \nu$.*
   - *(b) For all B, such that $B : \nu \in T$ and $A\Theta = B$ for some substitution Θ, we replace $B : \nu$ with $B : \nu \cap \mu$.*
   - *(c) Let $\mathcal{B} = \{\nu \mid B : \nu \in T \wedge (\exists \Theta) B\Theta = A\}$. We add $A : \mu \cap (\cap_{\nu \in \mathcal{B}}\{\nu\})$ to T.*
   - *(d) For each B such that $B : \nu \in T$ and $A\Theta_1 = B\Theta_2$ for some substitutions Θ_1 and Θ_2 we add $A\Theta_1 : \mu \cap \nu$ to T.*
   - *(e) If no clause for A had been added to T on previous steps, we add $A : \mu$ to T.*

An atomic update is not a "complete" b-cache update per se, but it will be at the core of a number of updates that we consider further. Informally, we can describe this process as follows: we check to see if T contains any formulas unifiable with A. If not, we just add $A : \mu$ to T. Otherwise, we look for formulas in T which have probabilities that can affect the probability of A, or vice versa (see example). Then we update probability ranges for all such formulas.

Example 13 Suppose our b-cache $T = \{p(a,Y) : [0.4, 0.7], p(b,Y) : [0.6, 0.9], p(X,a) : [0.5, 1]\}$ Below we show the results of $T \uplus_{at} A$ for a number of given atoms (we consider variables in all the formulas to be standardized apart).

A	$T \uplus_{at} A$
$p(X,Y) : [0.5, 0.95]$	$\{p(a,Y) : [0.5, 0.7], p(b,Y) : [0.6, 0.9], p(X,a) : [0.5, 0.95], p(X,Y) : [0.5, 0.95]\}$
$p(a,a) : [0.3, 0.6]$	$\{p(a,Y) : [0.4, 0.7], p(b,Y) : [0.6, 0.9], p(X,a) : [0.5, 1], p(a,a) : [0.4, 0.6]\}$
$p(b,Z) : [0.4, 0.8]$	$\{p(a,Y) : [0.4, 0.7], p(b,Y) : [0.6, 0.8], p(X,a) : [0.5, 1], p(b,a) : [0.5, 0.8]\}$

Atomic updates do not update annotated basic formulas that are not atomic, and hence the cache that results from an atomic update may not be maximally complete, i.e. it may be the case that $T \cup \{F : \mu\} \models G : \mu'$, but $(T \uplus_{at} F : \mu) \not\models G : \mu'$ for a non-atomic G. An alternative update strategy that propagates such updates is given below.

Definition 42 (Propagated Atomic Update - *pat*) Let T be a b-cache, F be a basic formula. A *Propagated Atomic Update* strategy (pat) is defined as follows:

1. *F is atomic.* $T \uplus_{pat} F : \mu = T \uplus_{at} F : \mu$.
2. *Let* $F = (F_1 *_\rho \ldots *_\rho F_m)$. $T \uplus_{pat} F : \mu = (\ldots(T \uplus_{at} F_1 : md_\rho(\mu)) \uplus_{at} \ldots) \uplus_{at} F_m : md_\rho(\mu))$

The Propagated Atomic Update strategy extends atomic updates onto complex formulas. Among the advantages of this strategy are its relative simplicity and the fact that it works for any bound b on a b-cache. However it is a rather weak strategy in the sense that because every updating formula gets broken into the atoms that constitute it, some information about the probability ranges of associated formulas is lost, i.e. it is not maximally complete. The following example demonstrates this fact.

Example 14 *Let* $T = \emptyset$ *and* $F = (p(a) \wedge_{inc} q(a)) : [0.3, 0.6]$. *By definition* $T' = T \uplus_{pat} F = \{q(a) : [0.3, 1], p(a) : [0.3, 1]\}$. *Now we have* $T'[(p(a) \wedge_{inc} q(a)] = [0.3 * 0.3, 1] = [0.09, 1] \supseteq [0.3, 0.6]$. *However, if the bound b is greater than 1, we could try to store F itself in T', and preserve information about its probability range.*

The above example suggests how the PAT strategy can be modified to be able to be more complete.

Definition 43 (Elementary b-cache update) *Let T be a b-cache, F be a basic formula. We define an elementary b-cache update strategy (denoted $\uplus_{eb}$ as follows:*

1. **Case 1.** $|F| = 1$. *(F is atomic).* $T \uplus_{eb} \{F : \mu\} = T \uplus_{at} \{F : \mu\}$
2. **Case 2.** $1 < |F| \leq b$. *Let* $F \equiv F_1 *_\rho \ldots *_\rho F_m$. *We proceed in a number of steps.*
   (a) *Let* $T' = T \uplus_{pat} F : \mu$
   (b) *Let* $\{\langle\nu_1, \Theta_1\rangle, \ldots \langle\nu_k, \Theta_k\rangle\} \subseteq T'[F]$ *be all pairs from* $T'[F]$, *s.t.,* $\nu \not\subseteq \mu$. *We proceed in steps. Let* $T^0 = T'$. *Consider* T^i $(0 \leq i < k)$ *constructed. We now construct* T^{i+1}.
      - *If* $F\Theta_{i+1} : \nu \in T'$ *we replace it with* $F : \mu \cap \nu$ *and declare the new b-cache to be the result of an update operation, i.e.* $T^{i+1} = (T^i - \{F\Theta_{i+1} : \nu\}) \cup \{F\Theta_{i+1} : \mu \cap \nu\}$
      - *If* $F\Theta_{i+1} : \nu \notin T'$ *we declare* $T^{i+1} = T^i \cup \{F\Theta_{i+1} : \mu \cap \nu\}$.

   (c) *Now we declare* $T \uplus_{eb} F : \mu = T^k$.
3. **Case 3.** $|F| > b$. *Let* $F = F_1 *_\rho \ldots *_\rho F_m$ *Let* $B_1, B_2, \ldots B_k$ *be all subformulas of F of size b. Then* $T \uplus_{eb} \{F : \mu\} = (T \uplus_{pat} F) \uplus_{eb} B_1 : md_\rho(\mu) \uplus_{eb} \ldots \uplus_{eb} B_k : md_\rho(\mu)$.

It is easy to notice that

Proposition 3 *(i)* $(\forall b > 0) eb \geq pat$ *(ii)* $e1 \equiv pat$

Proof.

(i). Let F be a basic formula and $\mu \in \mathcal{C}[0,1]$. Three cases are possible:

- F is atomic. In this case by definition of eb $T \uplus_{\mathrm{e}b} F:\mu = T \uplus_{at} F:\mu = T \uplus_{pat} F:\mu$.
- $1 < |F| \leq b$. In this case $T \uplus_{\mathrm{e}b} F:\mu$ is computed starting from $T' = T \uplus_{pat} F:\mu$ via a series of iterations which modify/add information about formulas unifiable with F. This means that for all basic formulas G **not** unifiable with F and for all intervals $\nu \in \mathcal{C}[0,1]$, if $G:\nu \in CN(T')$ (i.e., $T' \models G:\nu$) then $G:\nu \in CN(T \uplus_{\mathrm{e}b} F:\mu)$ (i.e., $T \uplus_{\mathrm{e}b} F:\mu \models G:\nu$).

  Let now H be a basic formula unifiable with F and let $T' \models H:\nu$. This means that there exists a substitution Θ such that $\langle \nu, \Theta \langle \in T'[F]$ and $H = F\Theta$. But then, by definition of elementary b-cache update strategy, $T \uplus_{\mathrm{e}b} F:\mu$ will contain formula $F\Theta : \mu\cap\nu = H:\mu\cap\nu$. Clearly, $\mu\cap\nu \subseteq \nu$ and therefore, $\{H:\mu\cap\nu\} \models H:\nu$, i.e., $T \uplus_{\mathrm{e}b} F:\mu \models H:\nu$.

  From the above we imply that $CN(T \uplus_{pat} F:\mu) \subset CN(T \uplus_{\mathrm{e}b} F:\mu)$.
- $|F| > b$.

  Let S be the set of all subformulas of F of size b. By definition of the elementary b-cache update we get:

  $T \uplus_{\mathrm{e}b} F:\mu = (T \uplus_{pat} F:\mu) \uplus_{\mathrm{e}b} S$.

  But by definition of an update strategy we get $CN(T \uplus_{pat} F:\mu) \subseteq CN((T \uplus_{pat} F:\mu) \uplus_{\mathrm{e}b} S = CN(S)$

(ii). To prove that e1 $\equiv$ *pat* we first note that for any formula F one of two possible cases holds:

- F is atomic. In this case $T \uplus_{\mathrm{e}1} F:\mu = T \uplus_{pat} F:\mu$ by definition.
- F is not atomic. In this case $|F| > 1$. Let $F = A_1 *_\rho \ldots *_\rho A_k$. By definition of elementary b-cache update $T \uplus_{\mathrm{e}1} F:\mu = (\ldots(T \uplus_{pat} F:\mu) \uplus_{\mathrm{e}1} A_1 : md_\rho(\mu)) \ldots \uplus_{\mathrm{e}1} A_k : md_\rho(\mu)) = (\ldots((T \uplus_{pat} F:\mu) \uplus_{pat} A_1 : md_\rho(\mu)) \ldots \uplus_{pat} A_k : md_\rho(\mu))$ (the latter equality holds, since all A_i are atomic). Since $(\forall 1 \leq i \leq k) F:\mu \models A_i : md_\rho(\mu)$ we conclude that $CN((T \uplus_{pat} F:\mu) \uplus_{pat} A_i : md_\rho(\mu)) \subseteq CN(T \uplus_{pat} F:\mu)$. On the other hand, by definition of an update strategy, we know that the reverse $(CN((T \uplus_{pat} F:\mu) \uplus_{pat} A_i : md_\rho(\mu)) \supseteq CN(T \uplus_{pat} F:\mu))$ is true. Therefore $CN((T \uplus_{pat} F:\mu) \uplus_{pat} A_i : md_\rho(\mu)) = CN(T \uplus_{pat} F:\mu)$ which implies that

$$(\ldots((T \uplus_{pat} F:\mu) \uplus_{pat} A_1 : md_\rho(\mu)) \ldots \uplus_{pat} A_k : md_\rho(\mu)) \equiv T \uplus_{pat} F:\mu.$$

$\square$

Elementary updates allow us to capture more information about the updating formula, but these updates still allow for the loss of information as is shown in the following example.

Example 15 *Let $T = \emptyset$ and $F = (A \wedge_{inc} B \wedge_{inc} C) : [0.4, 0.6]$ (A,B,C are ground atoms). Let $T' = T \uplus_{e3} F$. By definition above $T' = \{(A \wedge_{inc} B \wedge_{inc} C) : [0.4, 0.6], A:[0.4,1], B:[0.4,1], C:[0.4,1]\}$.*

We notice that $T'[(A \wedge_{inc} B)] = [0.16, 1]$. However, it is clear that $F \models (A \wedge_{inc} B) : [0.4, 1]$.

The following strategy is more complete than elementary b-cache updates, but is also more difficult to compute.

Definition 44 (Full b-cache update) *Let T be a b-cache, F be a basic formula.*
We define a full b-cache update strategy (denoted $\uplus_{fb}$) as follows:

1. **Case 1.** *$|F| = 1$. (F is atomic). $T \uplus_{fb} F:\mu = T \uplus_{at} F:\mu$*
2. **Case 2.** *Let $F \equiv F_1 *_\rho \ldots *_\rho F_m$, $m \leq b$. Let $B_1, \ldots B_k$ be all the subformulas of F of size $< m$. We declare $T \uplus_{fb} F:\mu = T \uplus_{eb} F:\mu \uplus_{eb} B_1 : md_\rho(\mu) \uplus_{eb} \ldots \uplus_{eb} B_k : md_\rho(\mu)$.*
3. **Case 3.** *$|F| > b$. Let $F = F_1 *_\rho \ldots *_\rho F_m$ Let $B_1, B_2, \ldots B_k$ be all subformulas of F of size $\leq b$. Then $T \uplus_{fb} F:\mu = (T \uplus_{pat} F) \uplus_{eb} B_1 : md_\rho(\mu) \uplus_{eb} \ldots \uplus_{eb} B_k : md_\rho(\mu)$.*

The following result tells us that the full b-cache update strategy is more complete than the elementary b-cache update strategy.

Proposition 4 *(i)* $(\forall b > 0) fb \geq eb$ *(ii)* $f1 \equiv e1 \equiv pat$
update strategy.

Proof. *(i).* Let $b > 0$, $T \in \mathcal{T}$, $F \in bf_{\mathcal{S}}(B_L)$ and $\mu \in \mathcal{C}[0,1]$. Three cases are possible:

1. $|F| = 1$ (i.e., F is atomic). In this case $T \uplus_{\mathrm{f}_b} F : \mu = T \uplus_{at} F : \mu = T \uplus_{\mathrm{e}_b} F : \mu$.
2. $1 < |F| \leq b$. Let $B_1 \ldots B_k$ be all *proper* subformulas of F. Then $T \uplus_{\mathrm{f}_b} \{F : \mu\} = T \uplus_{\mathrm{e}_b} F : \mu \uplus_{\mathrm{e}_b} B_1 : md_\rho(\mu) \uplus_{\mathrm{e}_b} \ldots \uplus_{\mathrm{e}_b} B_k : md_\rho(\mu)$. Then, by definition of an update strategy, $CN(T \uplus_{\mathrm{e}_b} F : \mu) \subseteq CN(T \uplus_{\mathrm{f}_b} \{F : \mu\})$.
3. $|F| > b$. Let Let $S_B = \{B_1 \ldots B_k\}$ be all subformulas of F; let $S_G = \{G_1, \ldots, G_s\}$ be all subformulas of F of size strictly less than b and $S_H = \{H_1, \ldots H_r\}$ be all subformulas of F of size of exactly b. Clearly $S_B = S_H \cup S_G$.

   Using the commutativity property of b-cache update strategies we can obtain the following: $T \uplus_{\mathrm{f}_b} F : \mu = (T \uplus_{pat} F : \mu) \uplus_{\mathrm{e}_b} B_1 : md_\rho(\mu) \uplus_{\mathrm{e}_b} \ldots \uplus_{\mathrm{e}_b} B_k : md_\rho(\mu) = (T \uplus_{pat} F) \uplus_{\mathrm{e}_b} S_H \uplus ebS_G = T \uplus_{\mathrm{e}_b} F : \mu \uplus_{\mathrm{e}_b} S_G$.

   From this we immediately conclude $CN(T \uplus_{\mathrm{e}_b} F : \mu) \subseteq CN(T \uplus_{\mathrm{f}_b} F : \mu)$.

(ii). Same as the proof of part (ii) of Proposition 3. □

As the reader may easily notice from the definitions, implementing atomic updates is easy, however, PAT is more efficient than the elementary b- cache strategies eb, which get less efficient as b gets larger – and finally, implementing the full b-cache strategies is hardest of all, with the efficiency of these updates degrading as b increases. This will become apparent from the examples shown in the next section.

5.5 Proof Procedure for HP-Programs with b-cache

In the previous section, we presented a query refutation procedure for hybrid probabilistic programs. We now modify that refutation procedure for the case of query resolution from an hp-program with b-cache.

Informally the desired resolution procedure works as follows. Initially we have query Q, program P, a b-cache update strategy u and our b-cache T is (initially) empty. On each resolution step, we select a basic formula $F : \mu$ from current query and perform a lookup for the probability range of this formula in our current b-cache. To do this we have to compute $T[F]$. Once $T[F]$ is computed we compare it to μ. In case $T[F] \subseteq \mu$ we consider the current resolution step done. Otherwise, we use refutation procedure described above to perform one resolution step. If we decide that this resolution step resulted in proving new basic formula, we use b-cache update strategy u to update the current b-cache with one or more newly proven formulas.

Definition 45 *Let* $Q \equiv \exists(F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n)$ *be an initial query to hp-program* P.
A <u>*b-cache supported initial query*</u> $\hat{Q}$ *is defined as follows: Let* $F_{i_1} : \mu_{i_1} \ldots F_{i_n} : \mu_{i_n}$ *be an arbitrary permutation of* $F_1 : \mu_1 \wedge \ldots \wedge F_n : \mu_n$. *Then* $\hat{Q} \equiv < (F_{i_1}\mu_{i_1}, \emptyset), \ldots, (F_{i_n} : \mu_{i_n}, \emptyset) >$.
Any initial b-cache supported query is a <u>*b-cache supported query.*</u>

It is clear from the definition above that one query to P of size n can generate $n!$ different b-cache supported queries.

Definition 46 *We define a* <u>*b-cache supported resolvent*</u> *and a* <u>*b-cache update procedure*</u> *simultaneously. Let* P *be an hp-program,* T *- a b-cache and* u *- a b-cache update strategy. Let* $\hat{Q} \equiv < (G_1 : \mu_1, S_1), \ldots, (G_m : \mu_m, S_m) >$, *where for each* $1 \leq i \leq m$, S_ρ *is a set (possibly empty) of annotated basic formulas (not necessarily ground). Two cases have to be considered:*

1. *There exists* $<\mu, \Theta> \in T[G_1]$*, such that,* $\mu \subseteq \mu_1$*. Let* $C \equiv G_1\Theta : \mu \longleftarrow$*. Then*

$$\hat{Q}' \equiv <(G_2\Theta : \mu_2, S_2\Theta), \ldots, (G_m\Theta : \mu_m, S_m\Theta)>$$

   is a b-cache supported resolvent of $\hat{Q}$ *and* C*.*
   A b-cache update procedure ϕ_u *for this case can be defined as follows:* $\phi_u(\hat{Q}, T, C, \Theta) = T \uplus_u S_1\Theta$*.*
2. *There is* **no** $<\mu', \Theta'> \in T[G_1]$*, such, that* $\mu' \subseteq \mu_1$*. In this case, let* $C \equiv G : \lambda \longleftarrow F_1 : \lambda_1 \wedge \ldots \wedge F_n : \lambda_n$*,* G_1 *unifies with* G *via max-gu* Θ *and* $\lambda \subseteq \mu_1$*. Let* $F_{i_1} : \lambda_{i_1} \ldots F_{i_n} : \lambda_{i_n}$ *be any arbitrary permutation of* $F_1 : \lambda_1 \wedge \ldots \wedge F_n : \lambda_n$*.*
   We define a b-cache supported resolvent of $\hat{Q}$*,*C *and* T *to be:*

$$\hat{Q}' \equiv <(F_{i_1}\Theta : \lambda_{i_1}, \emptyset), \ldots, (F_{i_n}\Theta : \lambda_{i_n}, S_1\Theta \cup \{G_1\Theta\}), (G_2\Theta : \mu_2, S_2\Theta), \ldots, (G_m\Theta : \mu_m, S_m\Theta)> .$$

   A b-cache update procedure for this case is defined as follows:
   (a) Body of C is **empty** .

$$\phi_u(\hat{Q}, T, C, \Theta) = T \uplus_u G_1\Theta : \lambda \uplus_u S_1\Theta$$

   (b) Body of C is **not** empty.

$$\phi_u(\hat{Q}, T, C, \Theta) = T$$

Definition 47 *Let* P *be an hp-program,* Q *- a query and* u *- a b-cache update strategy.*
A <u>*b-cache supported refutation*</u> *of* Q *from* P *via* HR_P *is a* finite *sequence*

$$<\hat{Q}_1, C_1, \Theta_1, T_1> \ldots <\hat{Q}_r, C_r, \Theta_r, T_r>$$

, where

- $\hat{Q}_1$ *is b-cache supported initial version of* Q*.*
- $T_1 = \emptyset$
- $\hat{Q}_r$ *is empty.*
- *for each* $1 \leq i \leq r$ *either* $P \vdash C_i$ *or* $T_i \vdash C_i$*.*
- *for each* $1 \leq i < r$*,* $\hat{Q_{i+1}}$ *is a b-cache supported resolvent of* $\hat{Q}_i$ *and* C_i *with max-gu* Θ_i*.*
- *for each* $1 \leq i < r$*,* $T_{i+1} = \phi_u(\hat{Q}_i, T_i, C_i, \Theta_i)$

Example 16 (2-cache supported hp-refutation with elementary update strategy) Let us return to the hp-program shown in Example 11 and the query considered there. We present below, a refutation using a 2-cache (i.e. $b = 2$) using the strategy $e2$, i.e. elementary 2-cache update. The reader will notice that using this strategy cuts the number of steps in the resolution by 3 steps, leading to an over 20% reduction in the length of a proof. Note that had we used a different update strategy, the reduction may have been different.

1. $Q_1 =<(a : [0.9, 1], \emptyset), (e : [1, 1], \emptyset)>$
   $T_1 = \emptyset; P \ni C_1 = a : [1, 1] \longleftarrow (b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1] \wedge f : [0.5, 0.9]$.
2. $Q_2 =<((b \wedge_{ind} c \wedge_{ind} d) : [0.25, 1], \emptyset), (f : [0.5, 0.9], \{a : [1, 1]\}), (e : [1, 1], \emptyset)>$
   $T_2 = \emptyset; P \vdash C_2 = (b \wedge_{ind} c \wedge_{ind} d) : [0.3, 1] \longleftarrow (c \wedge_{ind} d) : [0.3, 1]$.
3. $Q_3 =<((c \wedge_{ind} d) : [0.3, 1], \{(b \wedge_{ind} c \wedge_{ind} d) : [0.3, 1]\}), (f : [0.5, 0.9], \{a : [1, 1]\}), (e : [1, 1], \emptyset)>$
   $T_3 = \emptyset; P \vdash C_3 = (c \wedge_{ind} d) : [0.3, 1]) \longleftarrow$
4. $Q_4 =<(f : [0.5, 0.9], \{a : [1, 1]\}), (e : [1, 1], \emptyset)>$
   $T_4 = (T_3 \uplus_{2e} (c \wedge_{ind} d) : [0.3, 1])) \uplus_{2e} (b \wedge_{ind} c \wedge_{ind} d) : [0.3, 1] = \{c : [0.3, 1], d : [0.3, 1], b : [0.3, 1], (c \wedge_{ind} d) : [0.3, 1], (b \wedge_{ind} c) : [0.3, 1], (b \wedge_{ind} d) : [0.3, 1]\}$
   $P \vdash C_4 = f : [0.7, 0.9] \longleftarrow b : [1, 1]$.

5. $Q_5 =< (b:[1,1], \{f:[0.7,0.9], a:[1,1]\}), (e:[1,1],\emptyset) >$
$T_5 = T_4; T[b] = [0.3,1] \not\subseteq [1,1]$;
$P \ni C_5 = b:[1,1] \longleftarrow (c \wedge_{ind} d):[0.3,1]$.
6. $Q_6 =< ((c \wedge_{ind} d):[0.3,1], \{b:[1,1], f:[0.7,0.9], a:[1,1]\}, (e:[1,1],\emptyset) >$
$T_6 = T_5 = T_4; T_6[(c \wedge_{ind} d)] = [0.3,1] \subseteq [0.3,1]$
7. $Q_7 =< (e:[1,1],\emptyset) >$
$T_7 = (T_6 \uplus_{2e} (c \wedge_{ind} d):[0.3,1]) \uplus_{2e} \{b:[1,1], f:[0.7,0.9], a:[1,1]\} = \{c:[0.3,1], d:[0.3,1], b:[1,1], (c \wedge_{ind} d):[0.3,1], (b \wedge_{ind} c):[0.3,1], (b \wedge_{ind} d):[0.3,1], f:[0.7,0.9], a:[1,1]\}$
$P \ni C_7 = e:[1,1] \longleftarrow (b \wedge_{ind} c \wedge_{ind} d):[0.25,1] \wedge f:[0.5,0.1]$.
8. $Q_8 =< ((b \wedge_{ind} c \wedge_{ind} d):[0.25,1],\emptyset), (f:[0.5,0.1], \{e:[1,1]\}$
$T_8 = T_7; T_8[(b \wedge_{ind} c \wedge_{ind} d)] = [0.3,1] \subseteq [0.25,1]$
9. $Q_9 =< (f:[0.5,0.1], \{e:[1,1]\}$
$T_9 = T_8 = T_7; T_9[f] = [0.7,0.9] \subseteq [0.5,0.9]$
10. $Q_{10} = \Box$

The following two important results state that *irrespective* of which update strategy is used, b-cache supported hp-refutations are guaranteed to be sound and complete. (Completeness assumes that the program P is consistent). The proofs are straightforward, as we know that HR_P is sound and complete, and the b-cache supported hp-refutation via HR_P is just its conservative extension.

Theorem 12 (Soundness of b-cache supported hp-refutation via HR_P).
Let P be an hp-program, Q be an initial query, and $\uplus$ be any update strategy. If there exists a b-cache supported refutation via HR_P of $Q \equiv \exists(F_1:\mu_1 \wedge \ldots \wedge F_n:\mu_n)$ from P with the answer substitution Θ then $P \models \forall((F_1:\mu_1 \wedge \ldots \wedge F_n:\mu_n)\Theta)$.

Proof. Suppose

$$< \hat{Q}_1, C_1, \Theta_1, T_1 > \ldots < \hat{Q}_r, C_r, \Theta_r, T_r >$$

is a b-cache supported hp-refutation of Q w.r.t. HR_P. We proceed by induction on r.

Base Case ($r = 1$) In this case, as $T_1 = \emptyset$, it is immediate that $< \hat{Q}_1, C_1, \Theta_1 >$ is an HR_P-refutation of Q and the result follows immediately by the soundness theorem for HR_P-refutations.

Inductive Case ($r+1$) Consider the b-cache supported refutation

$$< \hat{Q}_2, C_2, \Theta_2, T_2 > \ldots < \hat{Q}_r, C_r, \Theta_r, T_r >$$

— it is easy to see that this may be viewed as a b-cache supported refutation of Q_2 from $P \cup T_2$. Hence, by the inductive hypothesis, $(P \cup T_2) \models (\forall)Q_2\Theta'$ where $\Theta' = \Theta_2 \cdots \Theta_r$.

But T_2 only contains $\phi_u(\hat{Q}_1, T_1, C_1, \Theta_1)$ and as u is a b-cache supported update policy, it follows that $CN((P \cup T_2)) \supseteq CN(\phi_u(\hat{Q}_1, T_1, C_1, \Theta_1))$ — hence, $\phi_u(\hat{Q}_1, T_1, C_1, \Theta_1) \models (P \cup T_2)$ and we are done. $\Box$

Theorem 13 (Completeness of b-cache supported hp-refutation).
Let P be a consistent hp-program which satisfies the fixpoint reachability conditions let Q' be a query, and $\uplus$ be any update strategy. Then, if $P \models \exists(Q')$ then there exists a b-cache supported hp-refutation of Q' from P via HR_P.

Proof. The proof is immediately obtained from the fact that every HR_P-refutation is a b-cache supported refutation — to see why, observe that the fourth bullet in the definition of a b-cache supported refutation requires either $P \vdash C_i$ or $T \vdash C_i$. The first case is the same as for HR_P refutations, and thus, every HR_P-refutation is a b-cache supported refutation. As HR_P-refutations are complete, so are b-cache supported refutations. $\Box$

Though we have not implemented the proof procedures described in this paper, two proof procedures for hybrid probabilistic programs have been implemented since the initial version of this paper was circulated. The first implementation, by Terrence Swift at the University of Maryland, uses the XSB system [28] to implement a large fragment of HPPs. The second is an implementation of a somewhat different fragment of HPPs by Stoffel at the University of Neuchatel in Switzerland. Stoffel in particular, has also suggested some performance-enhancing optimizations to the methods described here.

6 Related Work and Conclusions

Logic knowledge bases have been extended to handle fuzzy modes of uncertainty since the early 70's with the advent of the MYCIN and Prospector systems [11]. Shapiro was one of the first to develop results in fuzzy logic programming [30]. Baldwin [2] was one of the first to introduce *evidential* logic programming and a language called FRIL. Van Emden [36] was the first to provide formal semantical foundations for logic programs that was later extended by Subrahmanian [32] and then completely generalized in a succession of papers by Blair and Subrahmanian[4], and Fitting[13], Ginsberg [14], and applied to databases by Kifer and Li [17] and Kifer and Subrahmanian [18]. *All the above works did not obey the laws of probability.*

The first works in the area of probabilistic logic programming were due to Ng and Subrahmanian who, in a series of papers [24, 26], developed techniques for probabilistic logic programming under the assumption of *ignorance*. Their work built upon earlier work on probabilistic logics due to Fagin, Halpern and and others [12] and Nilsson [27].

In contrast, Kiessling and his group [15, 35, 29] have developed a framework called DUCK for reasoning with uncertainty. They provide an elegant logical axiomatic theory for uncertain reasoning in the presence of rules, and using the *independence assumption*.

Perhaps the most significant related work is the elegant recent work of Lakshmanan's group [21, 20, 22, 31]. Lakshmanan's group[22, 31] have been developing a parametric framework to represent varied probabilistic strategies in logic programs. This work, which was developed slightly ahead and independently of this paper[3], can express some of what we try to express here, though there is no support for basic formulas in the heads of rules, and hence there is also no need for decomposition functions. However, by a rather complex translation that significantly increases the size of a program, and by introducing specially programmed functions, they can express some hp-programs with atoms (not basic formulas !) in the heads in their syntax. Thus, the two approaches share a common intersection, but neither appears to subsume the other.

In addition, Lakshmanan and his colleagues complement our results with elegant query optimization results. Developing such results in the general setting of hp-programs remains a significant challenge, and will need to build upon the foundation laid by them in that arena. In contrast, our work offers a variety of cache-based query processing algorithms that complement their query optimization work, and merging the two offers much promise because query processing and optimization using materialized views (which is what a cache is) is well known to be very useful in enhancing performance[1].

There has been a substantial body of work on probabilistic extensions of relational databases, which we do not discuss here as their relation to logic programming is not immediate. For the sake of completeness, such works include[3, 6, 16, 19]. In particular, [19], among other things, introduces a set of operations on data which compute probabilities of compound events based on probabilities of simple events and the assumptions about the connections between the events. Our current work extends the framework

[3]Actually, the first paper that allows different conjunction and disjunction strategies to be incorporated into logic programs was [33].

of [19] onto logic programming by adding a notion of a "decomposition" function, which guides the computation of the probabilities of simple events based on the probability of the compound event.

In sum, our paper's goal was to provide a flexible probabilistic logic programming framework. Past approaches to logic programming with probabilities assumed that knowledge about all events in the real world represented by propositional symbols or predicate symbols took one single form – either we assumed ignorance of all dependences between such events (e.g. [24]) or we assumed independence (e.g. in most AI expert systems). In practice however, a probabilistic logic programming system must be flexible enough to allow the logic programmer to explicitly specify any domain specific knowledge he has about dependences (or lack thereof) between events. Our approach allows this, through the use of syntactic connectives that represent generalized conjunction/disjunction strategies. We have provided a formal model theoretic and fixpoint semantics for such hp-programs and shown that they are equivalent. We have further proposed three alternative execution paradigms for hp-programs.

In future work, we plan to build an hybrid probabilistic deductive database system that incorporates many of the ideas proposed in this paper. This system will be built on top of our ProbView[19] probabilistic relational database system. We hope to use this implementation, when complete, to experiment with different probabilistic query evaluation algorithms such as those described here, as well as probabilistic query optimization techniques that we hope to develop in the future.

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A Proof of Proposition 1

In this section we provide the complete proof of **Proposition 1**, which states that all strategies defined in Section 2.1 are indeed *coherent*, *conjunctive* or *disjunctive* p-strategies.

Proposition 1. *inc*, *igc* and *pcc* are continuous conjunctive coherent p-strategies. Similarly, *ind*, *igd*, *pcd* and *ncd* are continuous disjunctive coherent p-strategies.

Proof.

First we notice that **all** the composition functions under consideration satisfy the axiom of Separation. Indeed in the definition of every composition function $c_{inc}, c_{igc}, c_{pcc}, c_{ind}, c_{igd}, c_{pcd}, c_{ncd}$ the lower bound of the result is dependent only on the lower bounds of the arguments and similarly, the upper bound of the result depends only on the upper bounds of the arguments. Also, we notice that all composition functions mentioned above are continuous as all the functions that compute their lower and upper bounds are continuous in both arguments. Now, we prove the rest of the axioms for each individual strategy.

- *inc* is a conjunctive coherent p-strategy.
  - *inc* is a **conjunctive** p-strategy.
    Let us establish that *inc* satisfies all the axioms of a *conjunctive* p-strategy.
    1. Commutativity. $c_{inc}([a_1, b_1], [a_2, b_2]) = [a_1a_2, b_1b_2] = [a_2a_1, b_2, b_1] = c_{inc}([a_2, b_2], [a_1, b_1])$
    2. Associativity. $c_{inc}(c_{inc}([a_1, b_1], [a_2, b_2]), [a_3, b_3]) = c_{inc}([a_1a_2, b_1b_2], [a_3, b_3]) = [a_1a_2a_3, b_1b_2b_3] = [a_1(a_2a_3), b_1(b_2b_3)] = c_{inc}([a_1, b_1], [a_2a_3, b_2b_3]) = c_{inc}(([a_1, b_1], c_{inc}([a_2, b_2], [a_3, b_3])$
    3. Inclusion Monotonicity. Let $[a_1, b_1] \subseteq [a_3, b_3]$, i.e. $a_1 \geq a_3 \geq 0$ and $0 \leq b_1 \leq b_3$.
       $c_{inc}([a_1, b_1], [a_2, b_2]) = [a_1a_2, b_1b_2]$.
       $c_{inc}([a_3, b_3], [a_2, b_2]) = [a_3a_2, b_3b_2]$.
       $a_1 \geq a_3 \geq 0$ implies $a_1a_2 \geq a_3a_2$; $0 \leq b_1 \leq b_3$ implies $b_1b_2 \leq b_3b_2$, which in turn, means that $[a_1a_2, b_1b_2] \subseteq [a_3a_2, b_3b_2]$.
    4. Bottomline. $c_{inc}([a_1, b_1], [a_2, b_2] = [a_1a_2, b_1b_2]$. Since $0 \leq a_1, a_2 \leq 1$, $a_1a_2 \leq a_1$ and $a_1a_2 \leq a_2$, i.e. $a_1a_2 \leq \min(a_1, a_2)$.
       Similarly, since $0 \leq b_1, b_2 \leq 1$, $b_1b_2 \leq \min(b_1, b_2)$.
       This implies $[a_1a_2, b_1b_2] \leq_t [\min(a_1, a_2), \min(b_1, b_2)]$.
    5. Identity. $c_{inc}([a, b], [1, 1]) = [a \cdot 1, b \cdot 1] = [a, b]$
    6. Annihilator. $c_{inc}([a, b], [0, 0]) = [a \cdot 0, b \cdot 0] = [0, 0]$.
  - *inc* is a **coherent** p-strategy.
    We know that $d_{inc}([a, b]) = \{\langle[a_1, b_1], [a_2, b_2]\rangle | [a_1a_2, b_1b_2] = [a, b]\}$, i.e., $\langle[a_1, b_1], [a_2, b_2]\rangle \in d_{inc}([a, b])$ **iff** $c_{inc}([a_1, b_1], [a_2, b_2]) = [a, b]$, which means that *inc* is a coherent p-strategy.
- *igc* is a conjunctive coherent p-strategy.
  - *igc* is a **conjunctive** p-strategy.
    Let us establish that *igc* satisfies the axioms of *conjunctive* p-strategy.
    1. Commutativity. $c_{igc}([a_1, b_1], [a_2, b_2]) = [\max(0, a_1 + a_2 - 1), \min(b_1, b_2)] = [\max(a_2 + a_1 - 1, \min(b_1, b_2)] = c_{igc}([a_2, b_2], [a_1, b_1])$.
    2. Associativity. $c_{igc}(c_{igc}([a_1, b_1], [a_2, b_2]), [a_3, b_3]) = c_{igc}([\max(0, a_1+a_2-1), \min(b_1, b_2)], [a_3, b_3]) = [\max(0, \max(0, a_1 + a_2 - 1) + a_3 - 1), \min(\min(b_1, b_2), b_3)] =$
       $[\max(0, 0+a_3-1, a_1+a_2+a_3-2), \min(b_1, b_2, b_3)] = [\max(0, a_1+a_2+a_3-2), \min(b_1, b_2, b_3)] =$
       $[\max(0, a_1+\max(0, a_2+a_3-1)-1), \min(b_1, \min(b_2, b_3))] = c_{igc}([a_1, b_1], c_{igc}([a_2, b_2], [a_3, b_3]))$.
    3. Inclusion Monotonicity.
       Let $[a_1, b_1] \subseteq [a_3, b_3]$, i.e. $a_1 \geq a_3 \geq 0$ and $0 \leq b_1 \leq b_3$.
       $c_{igc}([a_1, b_1], [a_2, b_2]) = [\max(0, a_1 + a_2 - 1), \min(b_1, b_2)]$.

$c_{igc}([a_3, b_3], [a_2, b_2]) = [\max(0, a_3 + a_2 - 1), \min(b_3, b_2)]$.
Since $a_1 \geq a_3 \geq 0$, $a_1+a_2-1 \geq a_3+a_2-1$, i.e., $\max(0, a_1+a_2-1) \geq \max(0, a_3+a_2-1)$.
Since $0 \leq b_1 \leq b_3$, $\min(b_1, b_2) \leq \min(b_3, b_2)$. From this we obtain
$c_{igc}([a_1, b_1], [a_2, b_2]) = [\max(0, a_1+a_2-1), \min(b_1, b_2)] \subseteq [\max(0, a_3+a_2-1), \min(b_3, b_2)] = c_{igc}([a_3, b_3], [a_2, b_2])$.

4. Bottomline. $c_{igc}([a_1, b_1], [a_2, b_2]) = [\max(0, a_1 + a_2 - 1), \min(b_1, b_2)$.
   Clearly, $0 \leq \min(a_1, a_2)$. Also since $a_2 \leq 1$, $a_1 + a_2 - 1 \leq a_1$. Similarly, since $a_1 \leq 1$, $a_1 + a_2 - 1 \leq a_2$. The two inequalities allow us to deduce that $a_1 + a_2 - 1 \leq \min(a_1, a_2)$ and, therefore, $\max(0, a_1 + a_2 - 1) \leq \min(a_1, a_2)$.
   From this we obtain $[\max(0, a_1 + a_2 - 1), \min(b_1, b_2)] \leq_t [\min(a_1, a_2), \min(b_1, b_2)]$.
5. Identity. $c_{igc}([a, b], [1, 1]) = [\max(0, a + 1 - 1), \min(b, 1)] = [\max(0, a), \min(b, 1)] = [a, b]$.
6. Annihilator. $c_{igc}([a, b], [0, 0]) = [\max(0, a + 0 - 1), \min(b, 0)] = [0, 0]$.

– *igc* is a **coherent** p-strategy. Let $[a, b], [a_1, b_1], [a_2, b_2] \in \mathcal{C}[0, 1]$.
  * Let $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{igc}([a, b])$. We consider the following two possibilities for the relationships between a, a_1 and a_2:
    · $a = 0$ and $a_1 + a_2 \leq 1$. In this case $\max(0, a_1 + a_2 - 1) = 0 = a$.
    · $a > 0$ and $a_1 + a_2 - 1 = a$.

    As far as the relationships between b, b_1 and b_2 are concerned, by definition of d_{igc}, $b = b_1$ if $b_1 \leq b_2$ an $b = b_2$ if $b_2 \leq b_1$, which means that $b = \min(b_1, b_2)$.

    Combining our results together we obtain: $c_{igc}([a_1, b_1], [a_2, b_2]) = [\max(0, a_1 + a_2 - 1), \min(b_1, b_2)] = [a, b]$.
  * Let $c_{igc}([a_1, b_1], [a_2, b_2]) = [a, b]$. Then, we know that $[a, b] = [\max(0, a_1+a_2-1), \min(b_1, b_2)]$. This means that if $a = 0$, $\max(0, a_1 + a_2 - 1) = 0$, i.e., $a_1 + a_2 \leq 1$ and if $a > 0$ then $a = a_1 + a_2 - 1$.

    Similarly, $b = \min(b_1, b_2)$ implies, $b = b_1$ when $b_1 \leq b_2$ and $b = b_2$ when $b_2 \leq b_1$.

    This means that by definition of d_{igc} $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{igc}([a, b])$.

• *pcc* is a conjunctive coherent p-strategy.
  – *pcc* is a **conjunctive** p-strategy.
  Let us establish that *pcc* satisfies the axioms of *conjunctive* p-strategy.
  1. Commutativity. $c_{pcc}([a_1, b_1], [a_2, b_2]) = [\min(a_1, a_2), \min(b_1, b_2)] = [\min(a_2, a_1), \min(b_2, b_1) = c_{pcc}([a_2, b_2], [a_1, b_1])$
  2. Associativity. $c_{pcc}(c_{pcc}([a_1, b_1], [a_2, b_2]), [a_3, b_3]) = c_{pcc}([\min(a_1, a_2), \min(b_1, b_2)], [a_3, b_3]) = [\min(\min(a_1, a_2), a_3), \min(\min(b_1, b_2), b_3)] = [\min(a_1, a_2, a_3), (\min(b_1, b_2, b_3)] = [\min(a_1, \min(a_2, a_3)), \min(b_1, (\min(b_2, b_3))] = c_{pcc}([a_1, b_1], c_{pcc}([a_2, b_2], [a_3, b_3]))$.
  3. Inclusion Monotonicity.
     Let $[a_1, b_1] \subseteq [a_3, b_3]$, i.e. $a_1 \geq a_3 \geq 0$ and $0 \leq b_1 \leq b_3$.
     $c_{pcc}([a_1, b_1], [a_2, b_2]) = [\min(a_1, a_2), \min(b_1, b_2)]$.
     $c_{pcc}([a_3, b_3], [a_2, b_2]) = [\min(a_3, a_3), \min(b_1, b_2)]$.
     Since $a_1 \geq a_3 \geq 0$, $\min(a_1, a_2) \geq \min(a_3, a_3)$; since $0 \leq b_1 \leq b_3$, $\min(b_1, b_2)] \leq \min(b_1, b_2)]$. This implies $[\min(a_1, a_2), \min(b_1, b_2)] \subseteq [\min(a_3, a_3), \min(b_1, b_2)]$.
  4. Bottomline. $c_{pcc}([a_1, b_1], [a_2, b_2]) = [\min(a_1, a_2), \min(b_1, b_2)] \leq_t [\min(a_1, a_2), \min(b_1, b_2)]$ (since $\leq_t$ is reflexive).
  5. Identity. $c_{pcc}([a, b], [1, 1]) = [\min(a, 1), \min(b, 1)] = [a, b]$.
  6. Annihilator. $c_{pcc}([a, b], [0, 0]) = [\min(a, 0), \min(b, 0)] = [0, 0]$.

  – *pcc* is a **coherent** p-strategy.
  Let $[a, b], [a_1, b_1], [a_2, b_2] \in \mathcal{C}[0, 1]$.
  * Let $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{pcc}([a, b])$. Then either $a = a_1$ and $a_2 \geq a_1$ or $a = a_2$ and

$a_1 \geq a_2$. In either case $a = \min(a_1, a_2)$.

Similarly, since either $b = b_1$ and $b_2 \geq b_1$ or $b = b_2$ and $b_1 \geq b_2$, we get $b = \min(b_1, b_2)$.

Therefore $c_{pcc}([a_1, b_1], [a_2, b_2]) = [\min(a_1, a_2), \min(b_1, b_2)] = [a, b]$.

  * Let $c_{pcc}([a_1, b_1], [a_2, b_2]) = [a, b]$. Then $a = \min(a_1, a_2)$ and $b = \min(b_1, b_2)$. This means that either $a = a_1$ and $a_2 \geq a_1$ or $a = a_2$ and $a_1 \geq a_2$ and similarly, either $b = b_1$ and $b_2 \geq b_1$ or $b = b_2$ and $b_1 \geq b_2$. But this means that $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{pcc}([a, b])$.

- *ind* is a disjunctive coherent p-strategy.
  - *ind* is a **disjunctive** p-strategy.

    Let us establish that *ind* satisfies the axioms of *disjunctive* p-strategy.

    1. Commutativity. $c_{ind}([a_1, b_1], [a_2, b_2]) = [a_1+a_2-a_1a_2, b_1+b_2-b_1b_2] = [a_2+a_1-a_2a_1, b_2+b_1-b_2b_1] = c_{ind}([a_2, b_2], [a_1, b_1])$.
    2. Associativity. $c_{ind}(c_{ind}([a_1, b_1], [a_2, b_2]), [a_3, b_3]) = c_{ind}([a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2], [a_3, b_3]) = [a_1 + a_2 - a_1a_2 + a_3 - (a_1 + a_2 - a_1a_2)a_3, b_1 + b_2 - b_1b_2 + b_3 - (b_1 + b_2 - b_1b_2)b_3] = [a_1 + a_2 + a_3 - a_1a_2 - a_2a_3 - a_1a_3 + a_1a_2a_3, b_1 + b_2 + b_3 - b_1b_2 - b_2b_3 - b_1b_3 + b_1b_2b_3] = [a_1 + (a_2 + a_3 - a_2a_3) - a_1(a_2 + a_3 - a_2a_3), b_1 + (b_2 + b_3 - b_2b_3) - b_1(b_2 + b_3 - b_2b_3)] = c_{inc}([a_1, b_2], c_{inc}([a_2, b_2], [a_3, b_3]))$
    3. Inclusion Monotonicity.

       Let $[a_1, b_1] \subseteq [a_3, b_3]$, i.e. $a_1 \geq a_3 \geq 0$ and $0 \leq b_1 \leq b_3$.

       $c_{ind}([a_1, b_1], [a_2, b_2]) = [a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2]$.

       $c_{ind}([a_3, b_3], [a_2, b_2]) = [a_3 + a_2 - a_3a_2, b_3 + b_2 - b_3b_2]$.

       Since $a_1 \geq a_3 \geq 0$, we have $a_1 - a_1a_2 = a_1(1 - a_2) \geq a_3(1 - a_2) = a_3 - a_3a_2$ and therefore $a_1 + a_2 - a_1a_2 \geq a_3 + a_2 - a_3a_2$.

       Similarly, since $0 \leq b_1 \leq b_3$, we have $b_1 - b_1b_2 = b_1(1-b_2) \leq b_3(1-b_3) = b_3 - b_3b_2$ which in turn implies that $b_1 + b_2 - b_1b_2 \leq b_3 + b_2 - b_3b_2$.

       From this it follows that $[a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2] \subseteq [a_3 + a_2 - a_3a_2, b_3 + b_2 - b_3b_2]$.
    4. Bottomline. $c_{ind}([a_1, b_1], [a_2, b_2]) = [a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2]$.

       We have to show that $a_1 + a_2 - a_1a_2 \geq \max(a_1, a_2)$ and $b_1 + b_2 - b_1b_2 \geq \max(b_1, b_2)$.

       Indeed, since $0 \leq a_1, a_2 \leq 1$, $a_2 - a_1a_2 \geq 0$ and $a_1 - a_1a_2 \geq 0$. Therefore, $a_1 + (a_2 - a_1a_2) \geq a_1 + 0$ and $a_2 + (a_1 - a_1a_2) \geq a_2 + 0$, i.e. $a_1 + a_2 - a_1a_2 \geq \max(a_1, a_2)$.

       Similarly, since $0 \leq b_1, b_2 \leq 1$, $b_2 - b_1b_2 \geq 0$ and $b_1 - b_1b_2 \geq 0$. Therefore, $b_1 + (b_2 - b_1b_2) \geq b_1 + 0$ and $b_2 + (b_1 - b_1b_2) \geq b_2 + 0$, i.e. $b_1 + b_2 - b_1b_2 \geq \max(b_1, b_2)$.
    5. Identity. $c_{ind}([a, b], [0, 0]) = [a + 0 - a * 0, b + 0 - b * 0] = [a, b]$.
    6. Annihilator. $c_{ind}([a, b], [1, 1]) = [a + 1 - a * 1, b + 1 - b * 1] = [1, 1]$.
  - *ind* is a **coherent** p-strategy.

    Let $[a, b], [a_1, b_1], [a_2, b_2] \in \mathcal{C}[0, 1]$.

    * Let $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{ind}([a, b])$. Then $a_1 + a_2 - a_1a_2 = a$ and $b_1 + b_2 - b_1b_2 = b$, which means that $c_{ind}([a_1, a_2], [b_1, b_2]) = [a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2] = [a, b]$.
    * Let $c_{ind}([a_1, b_1], [a_2, b_2]) = [a, b]$. $c_{ind}([a_1, b_1], [a_2, b_2]) = [a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2]$, which means that $a = a_1 + a_2 - a_1a_2$ and $b = b_1 + b_2 - b_1b_2$. Therefore,$\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{ind}([a, b])$.
- *igd* is a disjunctive coherent p-strategy.
  - *igd* is a **disjunctive** p-strategy.

    Let us establish that *igd* satisfies the axioms of *disjunctive* p-strategy.

    1. Commutativity. $c_{igd}([a_1, b_1], [a_2, b_2]) = [\max(a_1, a_2), \min(1, b_1 + b_2)] =$
       $= [\max(a_2, a_1), \min(1, b_2 + b_1)] = c_{igd}([a_2, b_2], [a_1, b_1])$.
    2. Associativity. $c_{igd}(c_{igd}([a_1, b_1], [a_2, b_2]), [a_3, b_3]) = c_{igd}([\min(a_1, a_2), \max(1, b_1+b_2)], [a_3, b_3]) =$

$[\min(\min(a_1,a_2),a_3), \max(1, \max(1, b_1+b_2)+b_3)] = [\min(a_1,a_2,a_3), \max(1, b_1+b_2+b_3)] = [\min(a_1, \min(a_2,a_3), \max(1, b_1+\max(1, b_2+b_3))] = c_{igd}([a_1,b_1], c_{igd}([a_2,b_2],[a_3,b_3])$.

3. Inclusion Monotonicity.
   Let $[a_1,b_1] \subseteq [a_3,b_3]$, i.e. $a_1 \geq a_3 \geq 0$ and $0 \leq b_1 \leq b_3$.
   $c_{igd}([a_1,b_1],[a_2,b_2]) = [\max(a_1,a_2), \min(1, b_1+b_2)]$.
   $c_{igd}([a_3,b_3],[a_2,b_2]) = [\max(a_3,a_2), \min(1, b_3+b_2)]$.
   Since $a_1 \geq a_3 \geq 0$ we have $\max(a_1,a_2) \geq \max(a_3,a_2)$. Since $0 \leq b_1 \leq b_3$ we have $\min(1, b_1+b_2) \leq \min(1, b_3, b_2)$.
   This implies $[\max(a_1,a_2), \min(1, b_1+b_2)] \subseteq [\max(a_3,a_2), \min(1, b_3+b_2)]$.
4. Bottomline. $c_{igd}([a_1,b_1],[a_2,b_2]) = [\max(a_1,a_2), \min(1, b_1+b_2)]$.
   We have to show that $\min(1, b_1+b_2) \geq \max(b_1,b_2)$. This is clearly so, since, $b_1, b_2 \leq 1$, i.e. $\max(b_1,b_2) \leq 1$, and $b_1,b_2 \geq 0$, i.e., $b_1 \leq b_1+b_2$ and $b_2 \leq b_1+b_2$, which makes $\max(b_1,b_2) \leq b_1+b_2$, therefore yielding the desired result.
5. Identity. $c_{igd}([a,b],[0,0]) = [\max(a,0), \min(1,b)] = [a,b]$.
6. Annihilator. $c_{igd}([a,b],[1,1]) = [\max(a,1), \min(1, b+1)] = [1,1]$.

– *igd* is a **coherent** p-strategy.
  Let $[a,b],[a_1,b_1],[a_2,b_2] \in \mathcal{C}[0,1]$.
  * Let $\langle [a_1,b_1],[a_2,b_2]\rangle \in d_{igc}([a,b])$.
    Then either $a = a_1$ and $a_2 \leq a_1$ or $a = a_2$ and $a_1 \leq a_2$. In either case $a = \max(a_1,a_2)$.
    Also, either we have $b = 1$ and $b_1+b_2 \geq 1$ or $b < 1$ and $b_1+b_2 = b$. In either case $b = \min(1, b_1+b_2)$.
    But then, we get $c_{igc}([a_1,b_1],[a_2,b_2]) = [\max(a_1,a_2), \min(1, b_1+b_2)] = [a,b]$.
  * Let $c_{igc}([a_1,b_1],[a_2,b_2]) = [a,b]$.
    In this case $a = \max(a_1,a_2)]$ and $b = \min(1, b_1+b_2)$. This means that either $a = a_1$ and $a_2 \leq a_1$ or $a = a_2$ and $a_1 \leq a_2$ and also either $b = 1$ and $b_1+b_2 \geq 1$ or $b < 1$ and $b_1+b_2 = b$.
    But then, $\langle [a_1,b_1],[a_2,b_2]\rangle \in d_{igc}([a,b])$.

• *pcd* is a disjunctive coherent p-strategy.
  – *pcd* is a **disjunctive** p-strategy.
  Let us establish that *pcd* satisfies the axioms of *disjunctive* p-strategy.
  1. Commutativity. $c_{pcd}([a_1,b_1],[a_2,b_2]) = [\max(a_1,a_2), \max(b_1,b_2)] = [\max(a_2,a_1), \max(b_2,b_1)] = c_{pcd}([a_2,b_2],[a_1,b_1]$.
  2. Associativity. $c_{pcd}(c_{pcd}([a_1,b_1],[a_2,b_2]),[a_3,b_3]) = c_{pcd}([\max(a_1,a_2), \max(b_1,b_2)],[a_3,b_3]) = [\max(\max(a_1,a_2),a_3), \max(\max(b_1,b_2),b_3)] = [\max(a_1,a_2,a_3), \max(b_1,b_2,b_3)] = [\max(a_1, \max(a_2,a_3)), \max(b_1, \max(b_2,b_3))] = c_{pcd}([a_1,b_1], c_{pcd}([a_2,b_2],[a_3,b_3]))$.
  3. Inclusion Monotonicity.
     Let $[a_1,b_1] \subseteq [a_3,b_3]$, i.e. $a_1 \geq a_3 \geq 0$ and $0 \leq b_1 \leq b_3$.
     $c_{pcd}([a_1,b_1],[a_2,b_2]) = [\max(a_1,a_2), \max(b_1,b_2)]$.
     $c_{pcd}([a_3,b_3],[a_2,b_2]) = [\max(a_3,a_2), \max(b_3,b_2)]$.
     Since $a_1 \geq a_3 \geq 0$ we have $\max(a_1,a_2) \geq \max(a_3,a_2)$. Similarly, since $0 \leq b_1 \leq b_3$, we have $\max(b_1,b_2) \leq \max(b_3,b_2)$.
     This implies $[\max(a_1,a_2), \max(b_1,b_2)] \subseteq [\max(a_3,a_2), \max(b_3,b_2)]$.
  4. Bottomline. $c_{pcd}([a_1,b_1],[a_2,b_2]) = [\max(a_1,a_2), \max(b_1,b_2)] \leq_t [\max(a_1,a_2), \max(b_1,b_2)]$.
  5. Identity. $c_{pcd}([a,b],[0,0]) = [\max(a,0), \max(b,0)] = [a,b]$.
  6. Annihilator. $c_{pcd}([a,b],[1,1]) = [\max(a,1), \max(b,1)] = [a,a]$.

  – *pcd* is a **coherent** p-strategy.

Let $[a, b], [a_1, b_1], [a_2, b_2] \in \mathcal{C}[0, 1]$.

* Let $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{pcd}([a, b])$. Then either $a = a_1$ and $a_1 \geq a_2$ or $a = a_2$ and $a_2 \geq a_1$. In either case $a = \max(a_1, a_2)$. Similarly, we have either $b = b_1$ and $b_1 \geq b_2$ or $b = b_2$ and $b_2 \geq b_1$. In either case $b = \max(b_1, b_2)$.

  Therefore $c_{pcd}([a_1, b_1], [a_2, b_2]) = [\max(a_1, a_2), \max(b_1, b_2)] = [a, b]$.
* Let $c_{pcd}([a_1, b_1], [a_2, b_2]) = [a, b]$. In this case $a = \max(a_1, a_2)$ and $b = \max(b_1, b_2)$. This means that either $a = a_1$ and $a_1 \geq a_2$ or $a = a_2$ and $a_2 \geq a_1$ and also either $b = b_1$ and $b_1 \geq b_2$ or $b = b_2$ and $b_2 \geq b_1$. Therefore, $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{pcd}([a, b])$.

- *ncd* is a disjunctive coherent p-strategy.
  - *ncd* is a **disjunctive** p-strategy.

    Let us establish that *ncd* satisfies the axioms of *disjunctive* p-strategy.

    1. Commutativity. $c_{ncd}([a_1, b_1], [a_2, b_2]) = [\min(1, a_1 + a_2), \min(1, b_1 + b_2)] =$
    $[\min(1, a_2 + a_1), \min(1, b_2 + b_1)] = c_{ncd}([a_2, b_2], [a_1, b_1])$.
    2. Associativity. $c_{ncd}(c_{ncd}([a_1, b_1], [a_2, b_2]), [a_3, b_3]) = c_{ncd}([\min(1, a_1 + a_2), \min(1, b_1 + b_2)], [a_3, b_3]) =$
    $[\min(1, \min(1, a_1 + a_2) + a_3), \min(1, \min(1, b_1 + b_2) + b_3)] =$
    $[\min(1, a_1 + a_2 + a_3), \min(1, b_1 + b_2 + b_3)] =$
    $[\min(1, a_1 + \min(1, a_2 + a_3)), \min(1, b_1 + \min(1, b_2 + b_3))] = c_{ncd}([a_1, b_1], c_{ncd}([a_2, b_2], [a_3, b_3]))$.
    3. Inclusion Monotonicity.
    Let $[a_1, b_1] \subseteq [a_3, b_3]$, i.e. $a_1 \geq a_3 \geq 0$ and $0 \leq b_1 \leq b_3$.
    $c_{ncd}([a_1, b_1], [a_2, b_2]) = [\min(1, a_1 + a_2), \min(1, b_1 + b_2)]$.
    $c_{ncd}([a_3, b_1], [a_3, b_2]) = [\min(1, a_3 + a_2), \min(1, b_3 + b_2)]$.
    Since $a_1 \geq a_3 \geq 0$ we have $a_1 + a_2 \geq a_3 + a_2$ and therefore,
    $\min(1, a_1 + a_2) \geq \min(1, a_3 + a_2)$. Similarly, since $0 \leq b_1 \leq b_3$, we have
    $\min(1, b_1 + b_2) \leq \min(1, b_3 + b_2)$.
    This implies $[\min(1, a_1 + a_2), \min(1, b_1 + b_2)] \subseteq [\min(1, a_3 + a_2), \min(1, b_3 + b_2)]$.
    4. Bottomline. $c_{ncd}([a_1, b_1], [a_2, b_2]) = [\min(1, a_1 + a_2), \min(1, b_1 + b_2)]$.
    We need to show $\min(1, a_1 + a_2) \geq \max(a_1, a_2)$ and $\min(1, b_1 + b_2) \geq \max(b_1, b_2)$.

       Since, $a_1, a_2 \leq 1$, we can get $\max(a_1, a_2) \leq 1$, and since $a_1, a_2 \geq 0$, we have $a_1 \leq a_1 + a_2$ and $a_2 \leq a_1 + a_2$, which makes $\max(a_1, a_2) \leq a_1 + a_2$.

       Similarly, since, $b_1, b_2 \leq 1$, we can get $\max(b_1, b_2) \leq 1$, and since $b_1, b_2 \geq 0$, we have $b_1 \leq b_1 + b_2$ and $b_2 \leq b_1 + b_2$, which makes $\max(b_1, b_2) \leq b_1 + b_2$, therefore yielding the desired result.
    5. Identity. $c_{ncd}([a, b], [0, 0]) = [\max(a, 0), \max(b, 0)] = [a, b]$.
    6. Annihilator. $c_{ncd}([a, b], [1, 1]) = [\max(a, 1), \max(b, 1)] = [1, 1]$.
  - *ncd* is a **coherent** p-strategy.

    Let $[a, b], [a_1, b_1], [a_2, b_2] \in \mathcal{C}[0, 1]$.

    * Let $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{ncd}([a, b])$. Then we know that either $a = 1$ and $a_1 + a_2 \geq 1$ or $a < 1$ and $a_1 + a_2 = a$. This implies $a = \min(1, a_1 + a_2)$. Similarly, either $b = 1$ and $b_1 + b_2 \geq 1$ or $b < 1$ and $b_1 + b_2 = b$, which, in turn implies $b = \min(1, b_1 + b_2)$.

      From this it follows that $c_{ncd}([a_1, b_1], [a_2, b_2]) = [\min(1, a_1 + a_2), \min(1, b_1 + b_2)] = [a, b]$.
    * Let $c_{ncd}([a_1, b_1], [a_2, b_2]) = [a, b]$. Then $a = \min(1, a_1 + a_2)$ and $b = \min(1, b_1 + b_2)$. This means that if $a = 1$ then $a_1 + a_2 \geq 1$ and if $a < 1$ then $a_1 + a_2 = a$. Similarly, we get : if $b = 1$ then $b_1 + b_2 \geq 1$ and if $b < 1$ then $b_1 + b_2 = b$.

      From this we infer $\langle [a_1, b_1], [a_2, b_2] \rangle \in d_{ncd}([a, b])$. □