Data 401  MLE Practice

For each of the following, calculate the MLE analytically, if possible. In the notation below, $\bar{X} \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} X_i$, the mean of the values in our sample.

1. Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\lambda)$, with p.d.f.
   \[ p_\lambda(x) = \lambda e^{-\lambda x}. \]
   Show that $\hat{\lambda}_{\text{MLE}} = 1/\bar{X}$.
   Solution: Note the likelihood is:
   \[ \prod_{i=1}^{n} \lambda e^{-\lambda X_i}. \]
   Taking the log and simplifying goes as follows:
   \[ \log(\prod_{i=1}^{n} \lambda e^{-\lambda X_i}) = \sum_{i=1}^{n} \log(\lambda e^{-\lambda X_i}) = \sum_{i=1}^{n} (\log(\lambda) - \lambda X_i) = n \log(\lambda) - \sum_{i=1}^{n} \lambda X_i. \]
   Taking the derivative of that and setting it equal to 0 yields: $n/\lambda - \sum_{i=1}^{n} X_i = 0$
   Solving for lambda gives: $\hat{\lambda}_{\text{MLE}} = \frac{\sum_{i=1}^{n} X_i}{n}$, which clearly simplifies to: $\frac{1}{\bar{X}}$

2. Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda)$, with p.m.f.
   \[ p_\lambda(x) = e^{-\lambda} \frac{\lambda^x}{x!}. \]
   Show that $\hat{\lambda}_{\text{MLE}} = \bar{X}$.
   Solution:
   Likelihood: $\prod_{i=1}^{n} (e^{-\lambda} \frac{\lambda^{X_i}}{X_i!})$
   Logs lead to: $\log(\prod_{i=1}^{n} (e^{-\lambda} \frac{\lambda^{X_i}}{X_i!})) = \sum_{i=1}^{n} -\lambda + X_i \log(\lambda) - \log(X_i!) = -\lambda n + \log(\lambda) \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log(X_i!)$
   Taking the derivative and setting it to 0 gives: $-n + \frac{1}{\lambda} \sum_{i=1}^{n} X_i = 0$
   Simplifying gives: $\hat{\lambda}_{\text{MLE}} = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$

3. Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Normal}(\mu, 1)$, with p.d.f.
   \[ p_\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}. \]
   Show that $\hat{\mu}_{\text{MLE}} = \bar{X}$.
   Solution:
   Likelihood: $\prod_{i=1}^{n} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(X_i-\mu)^2} \right)$
   Log: $\sum_{i=1}^{n} \log(\frac{1}{\sqrt{2\pi}} + \log(e^{\frac{1}{2}(X_i-\mu)^2})) = \sum_{i=1}^{n} \log(\frac{1}{\sqrt{2\pi}}) - \frac{1}{2}(X_i - \mu)^2$
   Derivative (using the chain rule) and setting it equal to 0: $\sum_{i=1}^{n} (X_i - \mu) = 0$
   Simplifying: $\hat{\mu}_{\text{MLE}} = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$
4. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Normal}(0, \sigma^2)$, with p.d.f.

$$p_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} x^2}.$$ 

Show that $\hat{\sigma}^2_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} X_i^2$.

Solution:

Likelihood: $\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{X_i^2/(2\sigma^2)}$

Log Likelihood: $\sum_{i=1}^{n} \frac{1}{2} \log(2\pi\sigma^2) - \frac{X_i^2}{2\sigma^2} = \sum_{i=1}^{n} \frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\sigma^2) - \frac{X_i^2}{2\sigma^2}$

Derivative set to 0: $-\frac{n}{\sigma} + \frac{1}{\sigma} \sum_{i=1}^{n} X_i^2 = 0$

Simplification: $\hat{\sigma}^2_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} X_i^2$

5. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Weibull}(2, \lambda)$, with p.d.f.

$$p_{\lambda}(x) = 2\lambda^2 x e^{-(\lambda x)^2}.$$ 

Show that $\hat{\lambda}_{\text{MLE}} = \sqrt{n / \sum_{i=1}^{n} X_i^2}$.

Solution:

Likelihood: $\prod_{i=1}^{n} 2\lambda^2 X_i e^{-(\lambda X_i)^2}$

Log Likelihood: $\sum_{i=1}^{n} (\log(2) + 2 \log(\lambda) + \log(X_i) - (\lambda X_i)^2)$

Derivative: $0 = \sum_{i=1}^{n} (0 + \frac{2}{\lambda} + 0 - 2X_i(\lambda X_i)) = \sum_{i=1}^{n} \frac{2}{\lambda} - \sum_{i=1}^{n} 2X_i(\lambda X_i) = \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} X_i^2$

Solving for lambda: $\hat{\lambda}_{\text{MLE}} = (n/ \sum_{i=1}^{n} X_i^2)^{1/2}$

6. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Weibull}(\alpha, \lambda)$ with p.d.f.

$$p_{\lambda}(x) = \alpha \lambda (\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha}.$$ 

Show that $\hat{\lambda}_{\text{MLE}}(\alpha) = \sqrt{n / \sum_{i=1}^{n} X_i^\alpha}$.

Solution:

Likelihood: $\prod_{i=1}^{n} \alpha \lambda (\lambda X_i)^{\alpha-1} e^{-(\lambda X_i)^\alpha}$

Log Likelihood: $\sum_{i=1}^{n} (\log(\alpha) + \log(\lambda) + (\alpha - 1) \log(X_i) - (\lambda X_i)^\alpha) = \sum_{i=1}^{n} \log(\alpha) + \alpha \log(\lambda) + (\alpha - 1) \log(X_i) - (\lambda X_i)^\alpha$

Derivative: $\sum_{i=1}^{n} \alpha/\lambda - \alpha X_i (\lambda X_i)^{\alpha-1} = n\alpha/\lambda - \alpha \lambda^{\alpha-1} \sum_{i=1}^{n} X_i^\alpha = 0$

Simplifying: $\hat{\lambda}_{\text{MLE}}(\alpha) = (n/ \sum_{i=1}^{n} X_i^\alpha)^{1/\alpha}$
7. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Gamma}(2, \lambda)$, with p.d.f.

$$p_\lambda(x) = \lambda^2 x e^{-\lambda x}.$$ 

Show that $\hat{\lambda}_{\text{MLE}} = 2/\bar{X}$.

Solution:

Likelihood: $\prod_{i=1}^{n} \lambda^2 X_ie^{-\lambda X_i}$

Log Likelihood: $\sum_{i=1}^{n} 2 \log(\lambda) + \log(X_i) - \lambda X_i$

Derivative: $0 = \sum_{i=1}^{n} 2/\lambda - X_i = 2n/\lambda - \sum_{i=1}^{n} X_i$

Simplifying: $\hat{\lambda}_{\text{MLE}} = 2n/\sum_{i=1}^{n} X_i = 2/\bar{X}$

8. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Gamma}(\alpha, \lambda)$, with p.d.f.

$$p_\lambda(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.$$ 

Show that $\hat{\lambda}_{\text{MLE}}(\alpha) = \alpha/\bar{X}$. (Note: $\Gamma(\alpha)$ is a complicated function, but you don’t need to know anything about it to calculate this MLE!)

9. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Bernoulli}(\pi)$, with p.d.f.

$$p_\pi(x) = \pi^x (1 - \pi)^{1-x}$$

for $x = 0, 1$. Show that $\hat{\pi}_{\text{MLE}} = \bar{X}$ (which for this distribution, is the same as the proportion of ones).

10. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{NegativeBinomial}(r, \pi)$, with p.d.f.

$$p_\pi(x) = \binom{x-1}{r-1} \pi^r (1 - \pi)^{x-r}$$

for $x = r, r + 1, r + 2, \ldots$. Show that $\hat{\pi}_{\text{MLE}}(r) = r/\bar{X}$. 

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