Now let d be any element in $[b]_\equiv$. Then $b \equiv d$. The combination of $a \equiv b$, $b \equiv d$, and transitivity yields $a \equiv d$. That is, $d \in [a]_\equiv$. We have shown that every element in $[b]_\equiv$ is also in $[a]_\equiv$, so $[b]_\equiv \subseteq [a]_\equiv$. By a similar argument, we can establish that $[a]_\equiv \subseteq [b]_\equiv$. The two inclusions combine to produce the desired set equality. ■

Theorem 1.3.4

Let $\equiv$ be an equivalence relation over X. The equivalence classes of $\equiv$ partition X.

Proof. By Lemma 1.3.3, we know that the equivalence classes form a disjoint family of subsets of X. Let a be any element of X. By reflexivity, $a \in [a]_\equiv$. Thus each element of X is in one of the equivalence classes. It follows that the union of the equivalence classes is the entire set X. ■

1.4 Countable and Uncountable Sets

Cardinality is a measure that compares the size of sets. Intuitively, the cardinality of a set is the number of elements in the set. This informal definition is sufficient when dealing with finite sets; the cardinality can be obtained by counting the elements of the set. There are obvious difficulties in extending this approach to infinite sets.

Two finite sets can be shown to have the same number of elements by constructing a one-to-one correspondence between the elements of the sets. For example, the mapping

$$\begin{aligned} a &\longrightarrow 1 \\ b &\longrightarrow 2 \\ c &\longrightarrow 3 \end{aligned}$$

demonstrates that the sets $\{a, b, c\}$ and $\{1, 2, 3\}$ have the same size. This approach, comparing the size of sets using mappings, works equally well for sets with a finite or infinite number of members.

Definition 1.4.1

i) Two sets X and Y have the same cardinality if there is a total one-to-one function from X onto Y.

ii) The cardinality of a set X is less than or equal to the cardinality of a set Y if there is total one-to-one function from X into Y.

Note that the two definitions differ only by the extent to which the mapping covers the set Y. If the range of the one-to-one mapping is all of Y, then the two sets have the same cardinality.

The cardinality of a set X is denoted $card(\mathrm{X})$. The relationships in (i) and (ii) are denoted $card(\mathrm{X}) = card(\mathrm{Y})$ and $card(\mathrm{X}) \leq card(\mathrm{Y})$, respectively. The cardinality of X is said to be strictly less than that of Y, written $card(\mathrm{X}) < card(\mathrm{Y})$, if $card(\mathrm{X}) \leq card(\mathrm{Y})$ and $card(\mathrm{X}) \neq card(\mathrm{Y})$. The Schröder-Bernstein Theorem establishes the familiar relationship between $\leq$ and $=$ for cardinality. The proof of the Schröder-Bernstein Theorem is left as an exercise.

Theorem 1.4.2 (**Schröder-Bernstein**)
If $card(\mathrm{X}) \leq card(\mathrm{Y})$ and $card(\mathrm{Y}) \leq card(\mathrm{X})$, then $card(\mathrm{X}) = card(\mathrm{Y})$.

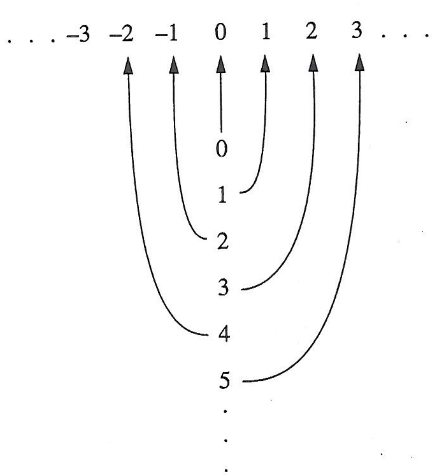
The cardinality of a finite set is denoted by the number of elements in the set. Thus $card(\{a, b\}) = 2$. A set that has the same cardinality as the set of natural numbers is said to be **countably infinite** or **denumerable.** Intuitively, a set is denumerable if its members can be put into an order and counted. The mapping f that establishes the correspondence with the natural numbers provides such an ordering; the first element is $f(0)$, the second $f(1)$, the third $f(2)$, and so on. The term **countable** refers to sets that are either finite or denumerable. A set that is not countable is said to be **uncountable**.

The set $\mathbf{N} - \{0\}$ is countably infinite; the function $s(n) = n + 1$ defines a one-to-one mapping from $\mathbf{N}$ onto $\mathbf{N} - \{0\}$. It may seem paradoxical that the set $\mathbf{N} - \{0\}$, obtained by removing an element from $\mathbf{N}$, has the same number of elements of $\mathbf{N}$. Clearly, there is no one-to-one mapping of a finite set onto a proper subset of itself. It is this property that differentiates finite and infinite sets.

Definition 1.4.3
A set is **infinite** if it has a proper subset of the same cardinality.

Example 1.4.1
The set of odd natural numbers is countably infinite. The function $f(n) = 2n + 1$ from Example 1.2.4 establishes the one-to-one correspondence between $\mathbf{N}$ and the odd numbers. □

A set is countably infinite if its elements can be put in a one-to-one correspondence with the natural numbers. A diagram of a mapping from $\mathbf{N}$ onto a set graphically illustrates the countability of the set. The one-to-one correspondence between the natural numbers and the set of all integers

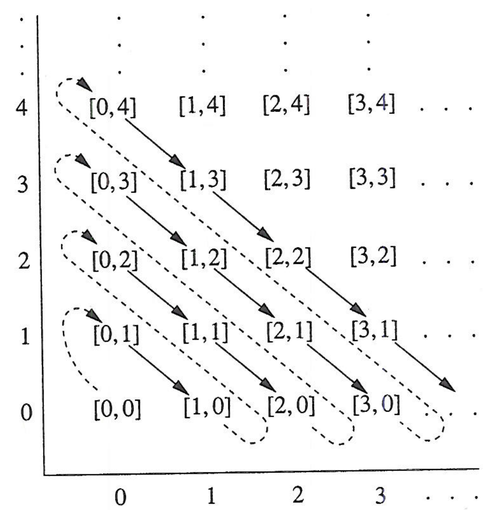


exhibits the countability of the set of integers. This correspondence is defined by the function

$$f(n) = \begin{cases} div(n, 2) + 1 & \text{if } n \text{ is odd} \\ -div(n, 2) & \text{if } n \text{ is even.} \end{cases}$$

Example 1.4.2

The points of an infinite two-dimensional grid can be used to show that $\mathbf{N} \times \mathbf{N}$, the set of ordered pairs of natural numbers, is denumerable. The grid is constructed by labeling the axes with the natural numbers. The position defined by the ith entry on the horizontal axis and the jth entry on the vertical axis represents the ordered pair $[i, j]$.



The elements of the grid can be listed sequentially by following the arrows in the diagram. This creates the correspondence

0	1	2	3	4	5	6	7	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
[0, 0]	[0, 1]	[1, 0]	[0, 2]	[1, 1]	[2, 0]	[0, 3]	[1, 2]	...

that demonstrates the countability of $\mathbf{N} \times \mathbf{N}$. The one-to-one correspondence outlined above maps the ordered pair $[i, j]$ to the natural number $((i + j)(i + j + 1)/2) + i$. □

The sets of interest in language theory and computability are almost exclusively finite or denumerable. We state, without proof, several closure properties of countable sets.

Theorem 1.4.4

i) The union of two countable sets is countable.

ii) The Cartesian product of two countable sets is countable.

iii) The set of finite subsets of a countable set is countable.

iv) The set of finite-length sequences consisting of elements of a nonempty countable set is countably infinite.

The preceding theorem indicates that the property of countability is retained under many standard set-theoretic operations. Each of these closure results can be established by constructing a one-to-one correspondence between the new set and a subset of the natural numbers.

A set is uncountable if it is impossible to sequentially list its members. The following proof technique, known as *Cantor's diagonalization argument,* is used to show that there is an uncountable number of total functions from N to N. Two total functions $f : \mathbf{N} \rightarrow \mathbf{N}$ and $g : \mathbf{N} \rightarrow \mathbf{N}$ are equal if they have the same value for every element in the domain. That is, $f = g$ if $f(n) = g(n)$ for all $n \in \mathbf{N}$. To show that two functions are distinct, it suffices to find a single input value for which the functions differ.

Assume that the set of total functions from the natural numbers to the natural numbers is denumerable. Then there is a sequence $f_0, f_1, f_2, \ldots$ that contains all the functions. The values of the functions are exhibited in the two-dimensional grid with the input values on the horizontal axis and the functions on the vertical axis.

	0	1	2	3	4	...
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	$f_0(4)$	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	...
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$f_3(4)$	...
f_4	$f_4(0)$	$f_4(1)$	$f_4(2)$	$f_4(3)$	$f_4(4)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Consider the function $f : \mathbf{N} \rightarrow \mathbf{N}$ defined by $f(n) = f_n(n) + 1$. The values of f are obtained by adding 1 to the values on the diagonal of the grid, hence the name diagonalization. By the definition of f, $f(i) \neq f_i(i)$ for every i. Consequently, f is not in the sequence $f_0, f_1, f_2, \ldots$. This is a contradiction since the sequence was assumed to contain all the total functions. The assumption that the number of functions is countably infinite leads to a contradiction. It follows that the set is uncountable.

Diagonalization is a general proof technique for demonstrating that a set is not countable. As seen in the preceding example, establishing uncountability using diagonalization is a proof by contradiction. The first step is to assume that the set is countable and therefore its members can be exhaustively listed. The contradiction is achieved by producing a member of the set that cannot occur anywhere in the list. No conditions are put on the listing of the elements other than that it must contain all the elements of the set. Producing a contradiction by diagonalization shows that there is no possible exhaustive listing of the elements and consequently that the set is uncountable. This technique is exhibited again in the following examples.

Example 1.4.3

A function f from N to N has a *fixed point* if there is some natural number i such that $f(i) = i$. For example, $f(n) = n^2$ has fixed points 0 and 1, while $f(n) = n^2 + 1$ has no fixed points. We will show that the number of functions that do not have fixed points is uncountable. The argument is similar to the proof that the number of all functions from N to N is uncountable, except that we now have an additional condition that must be met when constructing an element that is not in the listing.

Assume that the number of the functions without fixed points is countable. Then these functions can be listed $f_0, f_1, f_2, \ldots$. To obtain a contradiction to our assumption that the set is countable, we construct a function that has no fixed points and is not in the list. Consider the function $f(n) = f_n(n) + n + 1$. The addition of $n + 1$ in the definition of f ensures that $f(n) > n$ for all n. Thus f has no fixed points. By an argument similar to that given above, $f(i) \neq f_i(i)$ for all i. Consequently, the listing $f_0, f_1, f_2, \ldots$ is not exhaustive, and we conclude that the number of functions without fixed points is uncountable. □

Example 1.4.4

$\mathcal{P}(\mathbf{N})$, the set of subsets of N, is uncountable. Assume that the set of subsets of N is countable. Then they can be listed $N_0, N_1, N_2, \ldots$. Define a subset D of N as follows: For every natural number j,

$$j \in D \text{ if, and only if, } j \notin N_j.$$

By our construction, $0 \in D$ if $0 \notin N_0$, $1 \in D$ if $1 \notin N_1$, and so on. The set D is clearly a set of natural numbers. By our assumption, $N_0, N_1, N_2, \ldots$ is an exhaustive listing of the subsets of N. Hence, $D = N_i$ for some i. Is the number i in the set D? By definition of D,

$$i \in D \text{ if, and only if, } i \notin N_i.$$

But since $D = N_i$, this becomes

$$i \in D \text{ if, and only if, } i \notin D,$$

which is a contradiction. Thus, our assumption that $\mathcal{P}(\mathbf{N})$ is countable must be false and we conclude that $\mathcal{P}(\mathbf{N})$ is uncountable.

To appreciate the "diagonal" technique, consider a two-dimensional grid with the natural numbers on the horizontal axis and the vertical axis labeled by the sets N_0, N_1, $N_2, \ldots$. The position of the grid designated by row N_i and column j contains *yes* if $j \in N_i$. Otherwise, the position defined by N_i and column j contains *no*. The set D is constructed by considering the relationship between the entries along the diagonal of the grid: the number j and the set N_j. By the way that we have defined D, the number j is an element of D if, and only if, the entry in the position labeled by N_j and j is *no*. □