13. Let $G$ be a grammar in Greibach normal form and $M$ the PDA constructed from $G$. Prove that if $[q_0, u, \lambda] \vdash [q_1, \lambda, w]$ in $M$, then there is a derivation $S \Rightarrow uw$ in $G$. This completes the proof of Theorem 7.3.1.

14. Let $M$ be the PDA

$$Q = \{q_0, q_1, q_2\} \quad \delta(q_0, a, \lambda) = \{[q_0, A]\}$$
$$\Sigma = \{a, b\} \quad \delta(q_0, b, A) = \{[q_1, \lambda]\}$$
$$\Gamma = \{A\} \quad \delta(q_1, b, \lambda) = \{[q_2, \lambda]\}$$
$$F = \{q_2\} \quad \delta(q_2, b, A) = \{[q_1, \lambda]\}.$$

a) Give the state diagram of $M$.  
b) Give a set-theoretic definition of $L(M)$.  

c) Using the technique from Theorem 7.3.2, build a context-free grammar $G$ that generates $L(M)$.  
d) Trace the computation of $aabb$ in $M$.  

e) Give the derivation of $aabb$ in $G$.

15. Let $M$ be the PDA in Example 7.1.1.

a) Trace the computation in $M$ that accepts $bbcb$.  
b) Use the technique from Theorem 7.3.2 to construct a grammar $G$ that accepts $L(M)$.  
c) Give the derivation of $bbcb$ in $G$.

16. Theorem 7.3.2 presented a technique for constructing a grammar that generates the language accepted by an extended PDA. The transitions of the PDA pushed at most two variables onto the stack. Generalize this construction to build grammars from arbitrary extended PDAs. Prove that the resulting grammar generates the language of the PDA.

17. Use the pumping lemma to prove that each of the following languages is not context-free.

\[ a) \{a^k \mid k \text{ is a perfect square} \} \]
\[ b) \{a^ib^jc^j \mid i, j \geq 0 \} \]
\[ c) \{a^ib^ia^j \mid i \geq 0 \} \]
\[ d) \{a^ib^jc^k \mid 0 < i < j < k < zi \} \]
\[ e) \{ww^Rw \mid w \in \{a, b\}^*\} \]
\[ f) \text{The set of finite-length prefixes of the infinite string} \]
\[ \text{abaabaaabaaaaab} \ldots ba^nb \ldots \]

\[ \sqrt{18.} a) \text{Prove that the language } L_1 = \{a^ib^ja^j \mid i, j \geq 0 \} \text{ is context-free.} \]
\[ \sqrt{b) \text{Prove that the language } L_2 = \{a^ib^ja^j \mid i, j \geq 0 \} \text{ is context-free.} \]
\[ \sqrt{c) \text{Prove that } L_1 \cap L_2 \text{ is not context-free.} \]
19. a) Prove that the language \( L_1 = \{a^i b^j c^l d^j | i, j \geq 0\} \) is context-free.

b) Prove that the language \( L_2 = \{a^i b^j c^l d^k | i, j, k \geq 0\} \) is context-free.

c) Prove that \( L_1 \cap L_2 \) is not context-free.

20. Let \( L \) be the language consisting of all strings over \( \{a, b\} \) with the same number of \( a \)'s and \( b \)'s. Show that the pumping lemma is satisfied for \( L \). That is, show that every string \( z \) of length \( k \) or more has a decomposition that satisfies the conditions of the pumping lemma.

21. Let \( M \) be a PDA. Prove that there is a decision procedure to determine whether

a) \( L(M) \) is empty.

b) \( L(M) \) is finite.

c) \( L(M) \) is infinite.

22. A grammar \( G = (V, \Sigma, P, S) \) is called linear if every rule has the form

\[
\begin{align*}
A & \rightarrow u \\
A & \rightarrow uBv
\end{align*}
\]

where \( u, v \in \Sigma^* \) and \( A, B \in V \). A language is called linear if it is generated by a linear grammar. Prove the following pumping lemma for linear languages.

Let \( L \) be a linear language. Then there is a constant \( k \) such that for all \( z \in L \) with \( \text{length}(z) \geq k \), \( z \) can be written \( z = uvwx y \) with

i) \( \text{length}(uv) \leq k \),

ii) \( \text{length}(v) > 0 \), and

iii) \( uv^iwx^iy \in L \), for \( i \geq 0 \).

23. a) Construct a DFA \( N \) that accepts all strings in \( \{a, b\}^* \) with an odd number of \( a \)'s.

b) Construct a PDA \( M \) that accepts \( \{a^i b^i | i \geq 0\} \).

c) Use the technique from Theorem 7.5.3 to construct a PDA \( M' \) that accepts \( L(N) \cap L(M) \).

d) Trace the computations that accept \( aabab \) in \( N, M, \) and \( M' \).

24. Let \( G = (V, \Sigma, P, S) \) be a context-free grammar. Define an extended PDA \( M \) as follows:

\[
\begin{align*}
Q &= \{q_0\} \\
\delta(q_0, \lambda, \lambda) &= \{(q_0, S)\} \\
\Sigma &= \Sigma_G \\
\delta(q_0, \lambda, A) &= \{(q_0, w) | A \rightarrow w \in P\} \\
\Gamma &= \Sigma_G \cup V \\
\delta(q_0, a, a) &= \{(q_0, \lambda) | a \in \Sigma\} \\
F &= \{q_0\}
\end{align*}
\]

Prove that \( L(M) = L(G) \).

25. Complete the proof of Theorem 7.5.3.

26. Prove that the set

27. Let \( L \) be a context-free language. Show that \( \{a \} \cap L \) is context-free.

28. The notion of a context-free language

a) Prove that \( h(\) languages are

b) Use the result

c) Give an example that may be context-free

29. Let \( h : \Sigma^* \rightarrow \Sigma^* \) and \( \{w | h(w) \in L\} \) is closed under inverse

30. Use closure under languages are no

a) \( \{a^i b^i c^i d^i | i \geq 0\} \)

b) \( \{a^i b^i c^i d^i | i \geq 0\} \)

c) \( \{(ab)^i(bc)^i(c)\} \)

Bibliographic Notes

Pushdown automata were studied by the languages context-free languages by Evey [1963], and Schaefer [1960]. A solution to the pumping lemma is presented in Section 7.6.1. A stronger version of Theorem 7.6.6 proves...