Example 1.6.3

The shaded area in Figure 1.2(b) contains all the ordered pairs with second component 3, 4, 5, or 6. A recursive definition of this set, call it $X$, is given below.

i) Basis: $[0, 3]$, $[0, 4]$, $[0, 5]$, and $[0, 6]$ are in $X$.

ii) Recursive step: If $[m, n] \in X$, then $[s(m), n] \in X$.

iii) Closure: $[m, n] \in X$ only if it can be obtained from the basis elements by a finite number of applications of the operation in the recursive step.

The sequence of sets $X_i$ generated by this recursive process is defined by $X_i = \{(j, 3), (j, 4), (j, 5), (j, 6) | j = 0, 1, \ldots, i\}$.

1.7 Mathematical Induction

Establishing relationships between the elements of sets and operations on the sets requires the ability to construct proofs that verify the hypothesized properties. It is impossible to prove that a property holds for every member in an infinite set by considering each element individually. The principle of mathematical induction gives sufficient conditions for proving that a property holds for every element in a recursively defined set. Induction uses the family of nested sets generated by the recursive process to extend a property from the basis to the entire set.
Principle of Mathematical Induction  Let \( X \) be a set defined by recursion from the basis \( X_0 \) and let \( X_0, X_1, X_2, \ldots, X_i, \ldots \) be the sequence of sets generated by the recursive process. Also let \( P \) be a property defined on the elements of \( X \). If it can be shown that

i) \( P \) holds for each element in \( X_0 \),

ii) whenever \( P \) holds for every element in the sets \( X_0, X_1, \ldots, X_i \), \( P \) also holds for every element in \( X_{i+1} \),

then, by the principle of mathematical induction, \( P \) holds for every element in \( X \).

The soundness of the principle of mathematical induction can be intuitively exhibited using the sequence of sets constructed in the recursive definition of \( X \). Shading the circle \( X_i \) indicates that \( P \) holds for every element of \( X_i \). The first condition requires that the interior set be shaded. Condition (ii) states that the shading can be extended from any circle to the next concentric circle. Figure 1.3 illustrates how this process eventually shades the entire set \( X \).

The justification for the principle of mathematical induction should be clear from the preceding argument. Another justification can be obtained by assuming that conditions (i) and (ii) are satisfied but \( P \) is not true for every element in \( X \). If \( P \) does not hold for all elements of \( X \), then there is at least one set \( X_i \) for which \( P \) does not universally hold. Let \( X_j \) be the first such set. Since condition (i) asserts that \( P \) holds for all elements of \( X_0 \), \( j \) cannot be zero. Now \( P \) holds for all elements of \( X_{j-1} \) by our choice of \( j \). Condition (ii) then requires that \( P \) hold for all elements in \( X_j \). This implies that there is no first set in the sequence for which the property \( P \) fails. Consequently, \( P \) must be true for all the \( X_i \)’s, and therefore for \( X \).

An inductive proof consists of three distinct steps. The first step is proving that the property \( P \) holds for each element of a basis set. This corresponds to establishing condition (i) in the definition of the principle of mathematical induction. The second is the statement of the inductive hypothesis. The inductive hypothesis is the assumption that the property \( P \) holds for every element in the sets \( X_0, X_1, \ldots, X_n \). The inductive step then proves, using the inductive hypothesis, that \( P \) can be extended to each element in \( X_{n+1} \). Completing the inductive step satisfies the requirements of the principle of mathematical induction. Thus, it can be concluded that \( P \) is true for all elements of \( X \).

In Example 1.6.2, a recursive definition was given to generate the relation \( LT \), which consists of ordered pairs \([i, j]\) that satisfy \( i < j \). Does every ordered pair generated by the definition satisfy this inequality? We will use this question to illustrate the steps of an inductive proof on a recursively defined set.

The first step is to explicitly show that the inequality is satisfied for all elements in the basis. The basis of the recursive definition of \( LT \) is the set \( \{0, 1\} \). The basis step of the inductive proof is satisfied since \( 0 < 1 \).

The inductive hypothesis states the assumption that \( x < y \) for all ordered pairs \([x, y]\) \( \in \) \( LT \). In the inductive step we must prove that \( i < j \) for all ordered pairs \([i, j]\) \( \in \) \( LT \). The recursive step in the definition of \( LT \) relates the sets \( LT_{n+1} \) and \( LT_n \). Let \([i, j]\) be an ordered
pair in $LT_{n+1}$. Then either $[i, j] = [x, s(y)]$ or $[i, j] = [s(x), s(y)]$ for some $[x, y] \in LT_n$. By the inductive hypothesis, $x < y$. If $[i, j] = [x, s(y)]$, then

$$i = x < y < s(y) = j.$$ 

Similarly, if $[i, j] = [s(x), s(y)]$, then

$$i = s(x) < s(y) = j.$$
In either case, \( i < j \) and the inequality is extended to all ordered pairs in \( LT_{n+1} \). This completes the requirements for an inductive proof and consequently the inequality holds for all ordered pairs in \( LT \).

In the proof that every ordered pair \([i, j]\) in the relation \( LT \) satisfies \( i < j \), the inductive step used only the assumption that the property was true for the elements generated by the preceding application of the recursive step. This type of proof is sometimes referred to as simple induction. When the inductive step utilizes the full strength of the inductive hypothesis—that the property holds for all the previously generated elements—the proof technique is called strong induction. Example 1.7.1 uses strong induction to establish a relationship between the number of operators and the number of parentheses in an arithmetic expression.

**Example 1.7.1**

A set \( E \) of arithmetic expressions is defined recursively from symbols \( \{a, b\} \), operators \(+\) and \(-\), and parentheses as follows:

i) Basis: \( a \) and \( b \) are in \( E \).

ii) Recursive step: If \( u \) and \( v \) are in \( E \), then \( (u + v) \), \( (u - v) \), and \( -(v) \) are in \( E \).

iii) Closure: An expression is in \( E \) only if it can be obtained from the basis by a finite number of applications of the recursive step.

The recursive definition generates the expressions \( (a + b) \), \( (a + (b + b)) \), \( ((a + a) - (b - a)) \) in one, two, and three applications of the recursive step, respectively. We will use induction to prove that the number of parentheses in an expression \( u \) is twice the number of operators. That is, \( n_p(u) = 2n_o(u) \), where \( n_p(u) \) is the number of parentheses in \( u \) and \( n_o(u) \) is the number of operators.

Basis: The basis for the induction consists of the expressions \( a \) and \( b \). In this case, \( n_p(a) = 0 = 2n_o(a) \) and \( n_p(b) = 0 = 2n_o(b) \).

Inductive Hypothesis: Assume that \( n_p(u) = 2n_o(u) \) for all expressions generated by \( n \) or fewer iterations of the recursive step, that is, for all \( u \) in \( E_n \).

Inductive Step: Let \( w \) be an expression generated by \( n + 1 \) applications of the recursive step. Then \( w = (u + v) \), \( w = (u - v) \), or \( w = (v) \) where \( u \) and \( v \) are strings in \( E_n \). By the inductive hypothesis,

\[
\begin{align*}
    n_p(u) &= 2n_o(u) \\
    n_p(v) &= 2n_o(v).
\end{align*}
\]

If \( w = (u + v) \) or \( w = (u - v) \),

\[
\begin{align*}
    n_p(w) &= n_p(u) + n_p(v) + 2 \\
    n_o(w) &= n_o(u) + n_o(v) + 1.
\end{align*}
\]
Consequently,
\[2n_o(w) = 2n_o(u) + 2n_o(v) + 2 = n_p(u) + n_p(v) + 2 = n_p(w).\]

If \( w = (-v) \), then
\[2n_o(w) = 2(n_o(v) + 1) = 2n_o(v) + 2 = n_p(v) + 2 = n_p(w).\]

Thus the property \( n_p(w) = 2n_o(w) \) holds for all \( w \in E_{n+1} \) and we conclude, by mathematical induction, that it holds for all expressions in \( E \).

Frequently, inductive proofs use the natural numbers as the underlying recursively defined set. A recursive definition of this set with basis \( \{0\} \) is given in Definition 1.6.1. The \( n \)th application of the recursive step produces the natural number \( n \), and the corresponding inductive step consists of extending the satisfaction of the property under consideration from \( 0, \ldots, n \) to \( n + 1 \).

**Example 1.7.2**

Induction is used to prove that \( 0 + 1 + \cdots + n = n(n + 1)/2 \). Using the summation notation, we can write the preceding expression as
\[\sum_{i=0}^{n} i = n(n + 1)/2.\]

**Basis:** The basis is \( n = 0 \). The relationship is explicitly established by computing the values of each of the sides of the desired equality.

\[\sum_{i=0}^{0} i = 0 = 0(0 + 1)/2.\]

**Inductive Hypothesis:** Assume for all values \( k = 1, 2, \ldots, n \) that
\[\sum_{i=0}^{k} i = k(k + 1)/2.\]

**Inductive Step:** We need to prove that
\[\sum_{i=0}^{n+1} i = (n + 1)(n + 1 + 1)/2 = (n + 1)(n + 2)/2.\]
The inductive hypothesis establishes the result for the sum of the sequence containing \( n \) or fewer integers. Combining the inductive hypothesis with the properties of addition, we obtain

\[
\sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + (n + 1) \quad \text{(associativity of +)}
\]

\[= n(n + 1)/2 + (n + 1) \quad \text{(inductive hypothesis)}
\]

\[= (n + 1)(n/2 + 1) \quad \text{(distributive property)}
\]

\[= (n + 1)(n + 2)/2. \]

Since the conditions of the principle of mathematical induction have been established, we conclude that the result holds for all natural numbers. \( \square \)

Each step in the proof must follow from previously established properties of the operators or the inductive hypothesis. The strategy of an inductive proof is to manipulate the formula to contain an instance of the property applied to a simpler case. When this is accomplished, the inductive hypothesis may be invoked. After the application of the inductive hypothesis, the remainder of the proof often consists of algebraic manipulation to produce the desired result.

### 1.8 Directed Graphs

A mathematical structure consists of a set or sets, distinguished elements from the sets, and functions and relations on the sets. A distinguished element is an element of a set that has special properties that differentiate it from the other elements. The natural numbers, as defined in Definition 1.6.1, can be expressed as a structure \((\mathbb{N}, s, 0)\). The set \(\mathbb{N}\) contains the natural numbers, \(s\) is a unary function on \(\mathbb{N}\), and \(0\) is a distinguished element of \(\mathbb{N}\). Zero is distinguished because of its explicit role in the definition of the natural numbers.

Graphs are frequently used to portray the essential features of a mathematical entity in a diagram, which aids the intuitive understanding of the concept. Formally, a directed graph is a mathematical structure consisting of a set \(\mathbb{N}\) and a binary relation \(A\) on \(\mathbb{N}\). The elements of \(\mathbb{N}\) are called the nodes, or vertices, of the graph and the elements of \(A\) are called arcs or edges. The relation \(A\) is referred to as the adjacency relation. A node \(y\) is said to be adjacent to \(x\) when \([x, y] \in A\). An arc from \(x\) to \(y\) in a directed graph is depicted by an arrow from \(x\) to \(y\). Using the arrow metaphor, \(y\) is called the head of the arc and \(x\) the tail. The in-degree of a node \(x\) is the number of arcs with \(x\) as the head. The out-degree of \(x\) is the number of arcs with \(x\) as the tail. Node \(a\) in Figure 1.4 has in-degree two and out-degree one.

A path from a node \(x\) to a node \(y\) in a directed graph \(G = (\mathbb{N}, A)\) is a sequence of nodes and arcs \(x_0, [x_0, x_1], x_1, [x_1, x_2], x_2, \ldots, x_{n-1}, [x_{n-1}, x_n], x_n\) of \(G\) with \(x = x_0\) and \(y = x_n\). The node \(x\) is the initial node of the path and \(y\) is the terminal node. Each pair