Multitape Turing Machines
Multitape Turing Machines

Definition

A k-tape machine has k tapes and k independent tape heads. The machine reads the tapes simultaneously, but has only one state.

\[
\begin{array}{c}
\text{Tape 3} \\
\text{Tape 2} \\
\text{Tape 1}
\end{array}
\]

\[
q_i
\]
A multitape machine begins with the input in the standard position on Tape 1.
Multitape Turing Machines

- During each transition, a multitape machine may:
  1. Change the state.
  2. Write (potentially different) symbols to each tape.
  3. Independently reposition each tape head.

**Definition**

The transition function of a two-tape machine is of the form:

\[ \delta(q_i, x_1, x_2) = (q_j, y_1, d_1, y_2, d_2) \]

- \( q_i, q_j \in Q \) are the starting and ending states.
- \( x_1, x_2 \in \Gamma \) are scanned from Tape 1 and Tape 2.
- \( y_1, y_2 \in \Gamma \) are written to Tape 1 and Tape 2.
- \( d_1, d_2 \in \{L, R, S\} \) are the directions for Head 1 and Head 2.
Multitape Turing Machines

Example

Consider the following two-tape machine to accept \( \{ a^i b a^i \mid i \geq 0 \} \):

\[
\begin{align*}
q_0 &\xrightarrow{(B/B \ R, B/B \ R)} q_1 & q_2 &\xrightarrow{(B/B \ R, B/B \ R)} q_3 \\
q_1 &\xrightarrow{(B/B \ L)} q_2 & q_3
\end{align*}
\]

(a/a R, B/a R) \hspace{1cm} (a/a R, a/a L)

Example

Construct a multitape machine that, given an input \( BwB \) (on Tape 1), where \( w \in \{a, b\}^* \), produces the output \( BwwB \) (on Tape 1).
Converting Multitape to Multitrack Machines

**Theorem**

A language \( L \) is accepted by a multitape Turing machine if and only if it is accepted by a standard Turing machine.

- Note that a \( k \) tapes can be simulated by \((2k + 1)\) tracks.

**Example**

<table>
<thead>
<tr>
<th>Tape 2</th>
<th>Tape 1</th>
<th>Track 1</th>
<th>Track 2</th>
<th>Track 3</th>
<th>Track 4</th>
<th>Track 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Initial state: \( q_i \)
Converting Multitape to Multitrack Machines

To simulate a transition of a 2-tape $M$, a 5-track $M'$ must:

1. Find the ‘$X$’ on track 2; transition based on track 1.
2. Find the ‘$X$’ on track 4; transition based on track 3.

Note that we now know the symbols scanned from the “tapes”, so we know — at machine-design-time — the desired output of $\delta$.

3. Write to tracks 1 and 3; move the ‘$X$’s on tracks 2 and 4.

Example

Consider simulating $\delta(q_i, a, b) = (q_j, b, L, aR)$:

\[
\begin{align*}
(a/b \ L, \\
\qquad b/a \ R)
\end{align*}
\]

$q_i \quad \rightarrow \quad q_j$
Converting Multitape to Multitrack Machines

Example (cont.)

(where * indicates “any symbol”, and \(\{c\}\), “any symbol other than \(c\)"

\[
\begin{align*}
(\ast, \{X\}, \ast, \ast, \ast) / (\ast, X, \ast, \ast, \ast) & R \\
(\ast, \{X\}, \ast, \ast, \ast) & L \\
(b, X, \ast, \ast, \ast) / (b, X, \ast, \ast, \ast) & L \\
(\ast, \ast, \ast, \ast, \#) / (\ast, \ast, \ast, \ast, \#) & R \\
(\ast, \ast, \ast, \ast, \#) & L \\
(\ast, \ast, \ast, \ast, \ast) / (\ast, \ast, \ast, \ast, \ast) & R \\
(\ast, \ast, \ast, \ast, \ast) & L \\
(a, X, \ast, \ast, \ast) / (a, X, \ast, \ast, \ast) & L \\
(a, X, \ast, \ast, \ast) & R \\
(\ast, \ast, \ast, \ast, \#) / (\ast, \ast, \ast, \ast, \#) & R \\
(\ast, \ast, \ast, \ast, \#) & L \\
(\ast, \ast, b, X, \ast) / (\ast, \ast, b, X, \ast) & R \\
(\ast, \ast, b, X, \ast) & L
\end{align*}
\]