Improved Line Facility Location in Weighted Subdivisions
Extended Abstract

Ovidiu Daescu
University of Texas at Dallas
Richardson, Texas
daescu@utdallas.edu

Ka Yaw Teo
University of Texas at Dallas
Richardson, Texas
ka.teo@utdallas.edu

ABSTRACT

We present an improved solution for the line facility location problem in a planar weighted subdivision. Our solution makes use of a key observation that improves on some theoretical bounds, and can be much faster in practice when compared to the previously published solutions. The weighted region setup is a more realistic model for many facility location problems that arise in practical applications, and yet it has not received significant treatment due to the complexity associated with the model.

CCS CONCEPTS

• Theory of computation → Computational geometry;

KEYWORDS

Weighted regions, line, sum of distances, facility location

ACM Reference Format:

1 INTRODUCTION

We address the line facility location in a planar weighted subdivisions, a problem introduced in [3]. In the weighted subdivision setup, a two-dimensional workspace (the plane) is partitioned into convex regions, defined by a set \(V\) of \(m\) vertices and a corresponding set \(E\) of \(k = O(m)\) edges, and each of the convex regions is associated with a non-negative integer weight. In the defined workspace, a set of \(n\) points, \(A = \{a_1, a_2, \ldots, a_n\}\), is given, and the objective of the problem is to determine a straight line \(L\) that minimizes the sum of the weighted Euclidean distances from the given points in set \(A\) to line \(L\). To simplify the exposition, we assume that the regions are triangulated (Figure 1). The (Euclidean) weighted length, or length for short, of a line segment \(st\) is defined as the sum of the weighted lengths within each region intersected by \(st\), where the weighted length within a region is the Euclidean length multiplied by the weight of the region.

The planar facility location problem is one of the most studied problems in computer science and operations research (for example, see [5–7]). The hardness of the problem and the complexity of the algorithms used to solve it heavily depend on the definition of the objective function. One possible application under the min-sum weighted region formulation was described in [3]: the line facility \(L\) is an oil pipeline and the points in \(P\) are oil wells. The oil field is divided into (weighted) regions based on its characteristics, and from each well a spur pipeline is to be connected to \(L\) in a straight line. Another possible application, especially for a 3-dimensional (3D) version, is related to device placement for stimulating specific neural centers of the human brain. Currently, such stimulation is done mostly through electrode implants and electrical currents, and it is prone to significant side effects. Emerging technologies that involve focused, light based stimulation seem very promising, and a 3D extension of the work presented here or an application of the work in selected planes would optimize the placement of such devices to avoid or reduce side effects.

1.1 Related Work

In the (unweighted) plane, the problem of finding a line minimizing the sum of Euclidean distances from a set of \(n\) points to the line was solved by Megiddo and Tamir [7] more than three decades ago. They proved that an optimal line facility can be found in \(O(n^2 \log n)\) time by showing that there exists an optimal line that passes through at least two of the given points. Later, Imai et al. [6] proved that in \(L_1\)-metric the optimal solution can be computed in linear time.

There is a very little work done on the problem of line facility location in weighted regions [2, 3], as defined in this paper, primarily...
due to the difficulty of the problem. In [3], the authors showed that the weighted region version can be solved by dividing the problem into a number of $O(m^2n^2)$ 1-variable optimization subproblems. To this end, they proved a key property of the problem, specifically that there exists an optimal solution that passes through a set of critical points. The following lemma from [3] characterizes those critical points.

**Lemma 1.1.** If a line $L$ is optimal, then either $L$ passes through a point of $A$ or the Euclidean projection of a given point $a_i \in A$ onto line $L$ coincides with an edge $e_j$ of $E$ (e.g., $L = L_1$).

See Figure 2 for an illustration of the critical points.

In [1], the authors give a fundamental study of a somewhat related problem, called the optimal weighted link problem, where the goal is to minimize the weighted length of a line segment $st$ connecting two given regions, $R_i$ and $R_j$, of a weighted subdivision $R$ of the plane. They prove that the problem can be reduced to a number of $O(n^2)$ global optimization problems, which asks to minimize a 2-variable function over a convex domain. In [4], it has been shown that, in order to minimize those objective functions, $st$ must pass through a vertex of $R$. As a result, the problem reduces to solving $O(n)$ optimization problems at each vertex, where each objective function can be expressed as $f(x) = \sqrt{1+x^2}(d_0 + \sum_{i=1}^{m} \frac{a_i}{x+b_i})$, in which $d_0, a_i, b_i$ are constants, $t = O(n)$, $x$ is the slope of the line supporting $st$, and the feasible domain is a slope interval for the line passing through the vertex. A prune-and-search approach is also suggested in [4], which aims to compute an approximate solution by bisecting the slope interval and pruning out subproblems that cannot lead to an optimal solution.

### 1.2 Our Results

We provide a more careful analysis of the line facility location in weighted regions, including a different formalization of the objective function. Our main result is a key observation that, although simple in hindsight, has the potential of providing tremendous speedup in practice. Using Lemma 1.1, as shown in [3], the problem reduces to solving $O(m^2n^2)$ global optimization problems. For each such problem, the objective function is a 1-variable function expressed as a sum of $O(m+n)$ fractional terms. Our new observation should render most of the subproblems irrelevant in practice, keeping only those that satisfy a strong halving property. For those subproblems, one can find an approximate solution by using a prune-and-search strategy that solves all subproblems at once, as explained in [3]. Moreover, in theory, the observation leads to a reduction in the number of optimization subproblems to be considered from $O(m^2n^2)$ in [3] to $O(mn^2) - \alpha O(m)$ improvement. In practical applications, one would expect that $m$ is larger than $n$.

### 2 Problem Formulation and Properties

A two-dimensional workspace (the plane) is partitioned into convex regions defined by a set $V$ of $m$ vertices with a corresponding set $E$ of $k = O(m)$ edges, and each of the convex regions is associated with a non-negative integer weight. In the defined workspace, a set of $n$ points, $A = \{a_1, a_2, ..., a_n\}$, is given, and the objective of the problem is to determine a straight line $L$ that minimizes the sum of the weighted Euclidean distances from the given points in set $A$ to line $L$. In the following analysis, $\perp$ is used to denote the transpose, and $\perp$ to denote the orthogonal.

As illustrated in Figure 3, an arbitrary line $L$ can be defined as

$L(\alpha, \beta) = \{\alpha + \lambda \beta : \lambda \in \mathbb{R}\}$

where $\alpha, \beta \in \mathbb{R}^2$ and $\|\beta\|_2 = 1$. The length of the projection of a point $a_i$ to line $L$ is given by

$l_i(a_i, b_i) = \{a_i + rb_i : r \in \mathbb{R}\}$

where $a_i, b_i \in \mathbb{R}^2$ and $0 \leq r \leq 1$. Since the Euclidean projection point of $a_i$ onto $L$ is the point of $L$ at the minimum Euclidean distance from $a_i$, that is, $\alpha + \lambda \beta$ where $\lambda = \beta^\top (a_i - \alpha)$, $b_i$ can be expressed as

$b_i = \beta^\top (a_i - \alpha) \beta - (a_i - \alpha)$

An edge $e_j \in E$ is represented by

$e_j(c_j, d_j) = \{c_j + sd_j\}$

where $c_j, d_j \in \mathbb{R}^2$ and $0 \leq s \leq 1$. Thus, the intersection between an edge $e_j$ and a projection segment $l_i$, if any, is given by $a_i + r_{i,j}b_j$, where

$r_{i,j} = \frac{-(d_j^\top c_j)}{(d_j^\top b_i)}$
Note that $0 < t_{i,j} < 1$ must hold true for an edge $e_j$ that intersects with a projection segment $l_i$.

The objective function of the problem can then be defined as

$$f(L) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} w_i t_{i,j} \|b_i\|_2$$

where $t_{i,j} = r_{i,j} - r_{i,j-1}$ and $0 \leq t_{i,j} \leq 1$. Intuitively, $\|b_i\|_2$ is the length of the projection $l_i$ of $a_i$ onto $L$, $t_{i,j}$ is the fraction of $l_i$ that passes through the region of weight $w_j$, and $\{e_j : j = 1, 2, \ldots, k_i\}$ is the sequence of edges intersected by $l_i$, with $e_k$ being the same as $L$. By this notation, $w_k$ is the weight of the region where $l_i$ intersects with $L$.

A line $L$ divides the plane into two half-planes, $H^+$ and $H^-$ (Figure 4). Let $W, W^+, W^-, W^0$ be defined as, respectively,

- $W = \sum_{i=1}^{n} w_{k_i}$
- $W^+ = \sum_{i:a_i \in H^+} w_{k_i}$
- $W^- = \sum_{i:a_i \in H^-} w_{k_i}$
- $W^0 = \sum_{i:a_i \in \partial L} w_{k_i}$

Note that $W = W^+ + W^- + W^0$. We are now ready to state our main result.

**Lemma 2.1. (Halving Property)** If $L$ is optimal, then $W^+ \leq \frac{1}{2}W$ and $W^- \leq \frac{1}{2}W$.

**Proof.** We show the proof by contradiction. Suppose that $L$ is optimal, and $W^+ > \frac{1}{2}W$. Let $L'$ be a line parallel to $L$ at a small positive distance $\Delta \alpha$ away from $L$ in $H^+$, such that i) none of the points in $A$ changes from one side of $L$ to the other, and ii) the set of intersecting edges, $\{e_j : j = 1, 2, \ldots, k_i\}$, for each $l_i$ does not change (Figure 5). For simplicity of notation, let $d(a_i, L')$ be defined as the weighted Euclidean distance between a point $a_i$ and line $L'$. The objective function for $L'$ is

$$f(L') = \sum_{i:a_i \in H^+} d(a_i, L') + \sum_{i:a_i \in H^-} d(a_i, L') + \sum_{i:a_i \in \partial L} d(a_i, L')$$

where $d(a_i, L') = \Delta \alpha \sum_{i:a_i \in H^+} w_{k_i} - \Delta \alpha \sum_{i:a_i \in H^-} w_{k_i} - \Delta \alpha \sum_{i:a_i \in \partial L} w_{k_i}$.

According to our assumption, $W^+ > \frac{1}{2}W = \frac{1}{2}(W^+ + W^- + W^0)$ and subsequently $W^+ - W^- - W^0 > 0$. This leads to $f(L') < f(L)$, which implies that $L$ is not optimal. Hence, if $L$ is optimal, then $W^+ \leq \frac{1}{2}W$. A similar argument can be made to prove that if $L$ is optimal, then $W^- \leq \frac{1}{2}W$. \qed

### 2.1 Defining the solution space

We can now proceed in a manner similar to that in [3]. However, unlike [3], instead of evaluating all subproblems resulting from the intersections of curves in the dual arrangement, we only evaluate those subproblems for which the halving property holds. In practice, we expect the halving property will eliminate most of the subproblems from consideration. We detail the approach here, noting that we derive the optimization subproblems in a different way from [3].

In the subsequent discussion, a transition-point event refers to either i) a line $L$ passing through a given point $a_i \in A$, or ii) the Euclidean projection point of a given point $a_i \in A$ onto a line $L$ coincides with an edge $e_j$ of $E$.

From Lemma 1.1, if a line $L$ is optimal, then either $L$ passes through a given point $a_i$, or the Euclidean projection point of a given point $a_i$ onto $L$ coincides with an edge $e_j$, or both. This leads to two cases to be considered as follows.
When a transition-point event is encountered by line \( e_j \) at the intersection between edge \( e_j \) and the circle with a diameter defined by \( \alpha \) and \( a_i \).

**Case 1.** Suppose that \( L \) passes through a point \( a' \in A \). Given that \( L(\alpha, \beta) = \alpha + \beta \), if \( \alpha \) is set to \( a' \), then \( L \) becomes a function of only \( \beta \). As \( \beta \) varies, \( W^+ \) and \( W^- \) change in value only when a transition-point event is encountered. Thus, each transition-point event corresponds to an interval of \( \beta \) in which \( W^+ \) and \( W^- \) remain unchanged.

As \( \beta \) changes, line \( L \) with \( \alpha = a' \) can pass through any of the \( n-1 \) given points in \( A \setminus \{a'\} \), yielding \( n-1 \) intervals of \( \beta \). These \( \beta \)-intervals can be determined in sorted order for each point in \( A \) in an overall \( O(n^2) \) time by using arrangement sorting [3].

In addition, while \( \beta \) varies, the Euclidean projection point of any given point in set \( A \setminus \{a'\} \) onto line \( L \) with \( \alpha = a' \) can coincide with an edge \( e_j \) at most twice, and this yields at most \( 2k(n-1) \) \( \beta \)-intervals. These \( \beta \)-intervals can be determined by using Thales' theorem and intersections between a circle and a line, as shown in Figure 6 (i.e., check \( \binom{k}{2} \) circles against a total of \( k \) edges for intersections).

Thus, for a given point \( a_i \in A \), the number of \( \beta \)-intervals is at most \( n-1 + 2k(n-1) = 2k(n-1)+1 \). Altogether, for a total of \( n \) points in set \( A \), the number of \( \beta \)-intervals is in the order of \( O(n(2k+1)(n-1)) = O(kn^2) = O(mn^2) \). For each edge involved, the \( O(n^2) \) transition intervals can be sorted in \( O(n^2 \log n) \) time, giving a total of \( O(mn^2 \log n) \) time.

**Case 2.** Suppose that the Euclidean projection point of a point \( a' \in A \) onto line \( L \) coincides with an edge \( e' \). Given that the projection path of point \( a' \) to line \( L \) is given by \( L'(a', b') = a' + rb' \), the Euclidean projection point of \( a' \) onto \( L \) changes position along edge \( e' \), so does \( b' \) (Figure 7), and \( W^+ \) and \( W^- \) change in value only when a transition-point event is encountered by line \( L \). Thus, each transition-point event corresponds to an interval of \( b' \) in which \( W^+ \) and \( W^- \) remain constant. We can apply the halving property to decide whether to keep these intervals.

When the Euclidean projection point of \( a' \) onto \( L \) changes position along edge \( e' \), line \( L \) could pass through any of the \( n-1 \) points in \( A \setminus \{a'\} \), yielding \( n-1 \) \( b' \)-intervals. Note that any transition-point event due to the Euclidean projection point of any given point in \( A \setminus \{a'\} \) onto \( L \) coinciding with an edge \( e_j \) has already been accounted for. Thus, for a total of \( n \) points in set \( A \), along with a total of \( k \) edges, the number of \( b' \)-intervals is in the order of \( O(kn^2) \).

\( b' \)-intervals can be determined by using the same approach as in Case 1.

The resulting set of \( \beta \) and \( b \)-intervals can then be pruned by using the halving property, keeping only those ranges for which \( W^+ \leq \frac{1}{2}W \) and \( W^- \leq \frac{1}{2}W \).

With \( O(kn^2) = O(mn^2) \), we have an \( O(\log n) \) improvement over the previous result in [3], as summarized in the following lemma.

**Lemma 2.2.** The optimal line facility location \( L \) can be found by solving at most \( O(mn^2) \) 1-variable optimization subproblems. The feasible domains for the subproblems can be found in a total of \( O(mn^2 \log n) \) time.

### 2.2 Structure of the Objective Function

Recall that the objective function can be written as

\[
  f(L) = \sum_{i=1}^{n} \sum_{j=1}^{k} w_{ij} t_i j \beta ||b_i||_2
\]

where \( t_i j = r_i j - r_i j-1, 0 \leq t_i j \leq 1 \), and

\[
  b_i = \beta^T (a_i - \alpha) - (a_i - \alpha)
\]

In Case 1, \( \beta \) is the only variable in the objective function (for each \( \beta \)-interval). Since \( b_i \) is an affine-quadratic form in terms of \( \beta \), \( b_i \) is convex. Subsequently, \( \|b_i\|_2 \) is convex because it is a norm of a convex function \( b_i \). Both \( r_{i,j} \) and \( r_{i,j-1} \) are reciprocal functions of \( b_i \) and are positive, and hence they are convex. As a result, \( t_i j \), and thus \( f(L) \), is a difference of convex functions.

In Case 2, \( \beta \) is the only variable in the objective function (for each \( \beta \)-interval). If \( \alpha \) is set to the Euclidean projection point of a given point \( a_i \) onto line \( L \) coinciding with an edge \( e_j \), that is, \( \alpha = a_j + rb_j \), where \( r = 1 \) and \( \alpha = c_j + sd_j \), then \( b_i = c_j + sd_j - a_i \). This implies that \( b_i \) is an affine function of \( s \), where \( 0 \leq s \leq 1 \). Thus, \( b_i \) is convex. By following the same argument as in Case 1, \( f(L) \) is a difference of convex functions.

Notice that we have reached a similar conclusion for \( f(L) \) as in [3], but only derived differently. Unfortunately, in general, the difference of two convex functions can be neither convex nor concave, so it is yet unclear how to exploit this property of \( f(L) \).

The objective function only changes by a constant number of terms from one slope interval to the next, and thus it can be updated in constant time, as argued in [3]. Thus, we have:

**Lemma 2.3.** The \( O(mn^2) \) objective functions can be generated and maintained in an overall \( O(mn^2 \log n) \) time.


2.3 Optimization subproblems

In Case 1, for a given \( \beta \)-interval, which is associated with \( \alpha = a_i \in A \), the optimization problem is of the form

\[
f^* = \min_{\beta} f(L) \text{ subject to } \beta_l \leq \beta \leq \beta_h
\]

where \( \beta_l \) and \( \beta_h \) are the lower and upper bounds of the \( \beta \)-interval, respectively.

In Case 2, for each given \( b_i \)-interval, the optimization problem has the following form:

\[
f^* = \min_{b_i} f(L) \text{ subject to } b_{i,l} \leq b_i \leq b_{i,h}
\]

where \( b_{i,l} \) and \( b_{i,h} \) are the lower and upper bounds of the \( b_i \)-interval, respectively.

After solving the optimization problem for each interval that satisfies the halving property, the overall minimum \( f^* \) gives the optimal line \( L \) for the original problem.

In practice, rather than running an expensive and slow global optimization package, we suggest using the prune-and-search approximation introduced in [3]. As a reminder, that approach solves all optimization problems at once (thus suitable for parallel and GPU implementation), and has been shown to be very fast in practice.

3 A SPECIAL CASE

In this section, we address a special case where \( L \) is required to have an empty intersection with the convex hull of the points in \( A \).

Consider the problem of finding an optimal line \( L \) that lies outside the convex hull of \( A \). Let \( CH(A) = \{ p_i : i = 1, 2, \ldots, n' \} \) denote the convex hull of \( A \), and assume that the weight of the complement of \( CH(A) \) is 1 (i.e., the weighted regions lie within \( CH(A) \)). We have the following lemma (the proof is simple and thus omitted).

Lemma 3.1. The optimal line \( L \) must pass through a vertex of \( CH(A) \).

Notice that the lemma above virtually eliminates Case 2 of the general version from consideration. Another practical advantage of the special case is that the size \( n' \) of \( CH(A) \) could be considerably smaller than the size \( n \) of \( A \).

For each given vertex \( p_i \) of \( CH(A) \), which is associated with a \( \beta \)-interval, the optimization problem is of the form

\[
f^* = \min_{\beta} f(L: \alpha = p_i) \text{ subject to } \beta_l \leq \beta \leq \beta_h
\]

where

\[
\beta_l = \frac{p_i - p_{i-1}}{\|p_i - p_{i-1}\|} \quad \text{and} \quad \beta_h = \frac{p_{i+1} - p_i}{\|p_{i+1} - p_i\|}
\]

Given that \( n' \leq n \), the number of \( \beta \)-intervals is in the order of \( O(n) \). The optimal line for the problem is given by the pair \((\alpha = p_i, \beta)\) that produces the overall minimum \( f^* \).

We can generalize Lemma 3.1 as follows. Let \( P \) be a convex polygon enclosing \( A \), and assume that the weight of the complement of \( P \) is 1 (i.e., the weighted regions lie within \( P \)).

Corollary 3.2. The optimal line \( L \) that has an empty intersection with the interior of \( P \) must pass through a vertex of \( P \).

This result extends to 3D and higher dimensions. Specifically, let \( P \) be a convex polyhedron enclosing a set \( A \) of \( n \) points distributed within a weighted partition of the \( d \)-dimensional Euclidean space, and assume that the weight of the complement of \( P \) is 1 (i.e., the weighted regions lie within \( P \)). Let \( L \) be a hyperplane that minimizes the sum of weighted Euclidean distances from the points in \( A \) to \( L \). We have the following:

Corollary 3.3. The optimal hyperplane \( L \) must pass through a vertex of \( P \).

Obviously, all the results in this section apply to the unweighted case.

Relevance of the special case. In the context of noninvasive brain electrode placement on a (human) subject, minimizing the region-weighted distances between a set of points and a line can be thought of as maximizing the current flow along the orthogonal paths between a set of targeted structures (represented as points) and a line electrode in a chosen plane section.

An orthogonal path (between a point \( a_i \) and a line \( L \) that passes through \( k_i \) weighted regions (various brain structures) can be perceived as an electrical circuit in which \( k_i \) resistors are arranged in a series fashion. The electrical resistance of a brain region can be represented by using a non-negative weight. Thus, in this model, given a constant potential difference, the objective of minimizing the sum of region-weighted distances from a set of target points to a line electrode is equivalent to maximizing the sum of current flows between the target points and the line electrode.

4 CONCLUSION

We have presented improved results for the planar line facility location problem in weighted subdivisions, both as theoretical results and as practical expectations. We have also extended our results to a few special cases, including in 3D and higher dimensions. However, for the \( \mathbb{R}^d \) case with \( d \geq 3 \), we have replaced line \( L \) with a hyperplane. It is not clear how to extend our results to 3D when \( L \) remains a line, even for the unweighted case. The 3D version is especially appealing due to its applications in non-invasive brain neuromodulation.

REFERENCES